ELEMENTS OF GEOMETRY

Approved as a Text-Book for Higher Classes of High English Schools in Bengal. (Vide Calcutta Gazette, 13th Nov., 19[illegible]0)

ELEMENTS OF GEOMETRY

PARTS I—III

(Covering Books I—IV of *Euclid's Elements*)

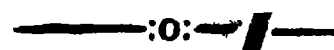
BY

NARES CHANDRA GHOSE, M.A.

PROFESSOR OF MATHEMATICS,

PRESIDENCY COLLEGE,

CALCUTTA.



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PREFACE.

This is a new book and I trust there is room for it. Its aims are to make this fascinating subject of Geometry more alive to the Indian boy, to give him an insight into the fundamental ideas which underlie the subject, and to present the materials on which he has to work, in a logical sequence.

It will be observed that I have incorporated in this book ideas of other writers : in particular I must mention the following :

W. K., Clifford—'*The Common Sense of the Exact Sciences*' (Kegan Paul Trench & Co.),

Benchara Branford—'*A Study of Mathematical Education*' (Clarendon Press.)

A. T. Simmons and Hugh Richardson—'*An Introduction to Practical Geography*' (Macmillan & Co. Ltd.)

David Mair—'*A School Course of Mathematics*' (Clarendon Press).

I hereby express my grateful thanks for the help these stimulating books have given me.

Nares Chandra Ghose.

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PART II—The Circle

PART III—Rectangles and Squares.

ELEMENTS OF GEOMETRY

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CHAPTER I.—INTRODUCTION.

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POINTS, LINES, SURFACES AND SOLIDS.

1. Points, lines, surfaces.

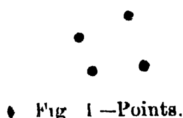


Fig. 1—Points.

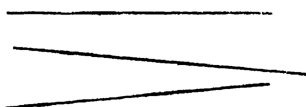


Fig. 2—Straight lines.



Fig. 3—Curved line

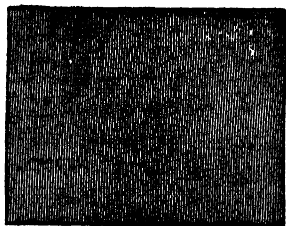


Fig. 4 (a)—Flat surface.

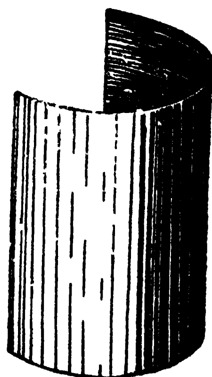


Fig. 4 (b)—Curved surface.

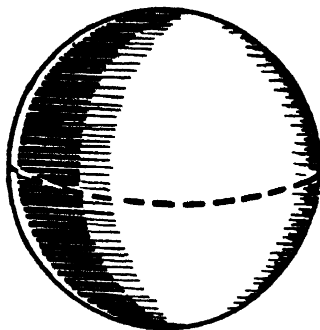


Fig. 4 (c)—Curved surface

A *thick line* (Fig. 5) is really a surface bounded by two lines (top and bottom).



Fig. 5—Thick line.

2. Why lines must not be thick and points must not be big.[1]—What is the size of the figure marked off by the four

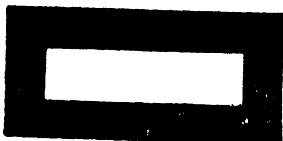


Fig 6

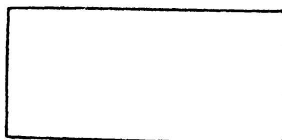


Fig. 7

thick lines (Fig. 6)? Here arises an ambiguity, shall we take the outer lines or the inner lines for the boundary? The ambiguity will disappear if we draw the lines so *thin* that they would not take up any appreciable room (Fig. 7).

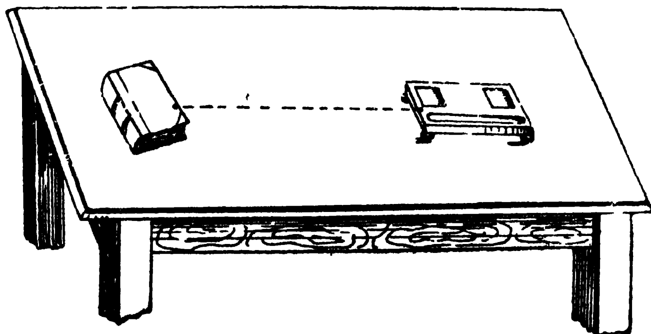


Fig 8

What is the distance from the book to the inkstand lying on the table (Fig 8)? To be accurate it is necessary to state from which

1. This mode of presentation is due to Branford, '*Mathematical Education*. (Clarendon Press, Oxford).

part of the book to which *part* of the inkstand. These parts should be marked by *points* which must be *so small as not to occupy any appreciable room*

For, if the points were big (Fig. 9), a number of straight lines could be drawn from one point to the other; the lengths of these

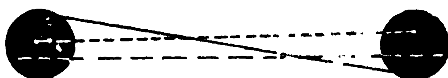


Fig. 9

lines would be different and the distance between the two points could not be precisely stated.

It will thus be more convenient if we use the terms *'line'* and *'point'* in a *restricted* sense. *A line must be thin (without breadth,* if we could draw it so) *and a point is just a mark, a dot not taking up any appreciable room.*

When points and lines are taken in this restricted sense we shall have :—

(i) **only one straight line passing through two given points** [Fig. 10 (*a*)].



Fig. 10 (a)

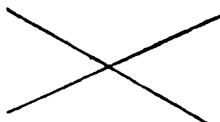


Fig. 10 (b)

(ii) **only one point common to two given straight lines** [Fig. 10 (*b*)].

Thus a point may be marked as an intersection of two lines (Fig. 11) which may be very short.

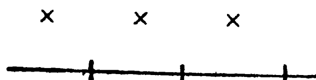


Fig. 11

What in ordinary language we call a thick line or a big point is not really a line or a point, but surfaces occupying more or less room.

3. Straight lines. How they can be drawn.

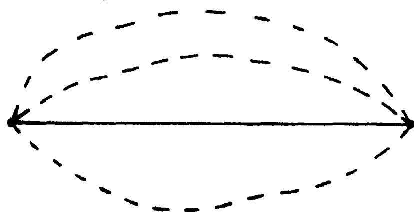


Fig. 12

Through two given points any number of lines may be drawn (Fig. 12), but only *one of them is straight*, and that one is the *shortest path* from one point to the other.

If you fold a piece of paper (Fig. 13) properly you get a crease which is a straight line. By holding the paper against the blackboard and guiding a chalk along the crease you trace a straight line on the board.

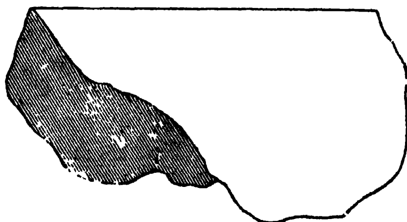


Fig. 13

If you stretch a piece of string or thread you get a straight line.

It would be better if you could have a stronger material than paper to guide your pencil. Take a thin strip of wood and trace a straight line on it with the help of a creased paper or a piece of string. Sawing along this line you get two pieces. Take one of the pieces and plane the sawn face to remove the roughness. You thus obtain a *straight* edge and guiding your pencil along it you can rule straight lines on a sheet of paper. The instrument itself may be called a

straight-edge or ruler.[1] It could be prepared from a sheet of metal as well.

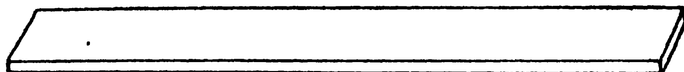


Fig. 14—A Ruler.

Sometimes a round ruler is used (specially in offices) for *ruling* lines. With good practice you can learn to draw straight lines freehand without the use of a ruler.

How would you mark a white straight line on a long plank of wood ?

How would you draw the straight line joining two given points on a sheet of paper ?

Hints on the tracing of lines. "Pencils should be neither too hard nor too soft; with the former, grooves are frequently made in the paper; while with the latter, it is difficult to draw lines sufficiently fine for accurate work, as well as to preserve a uniform thickness of line throughout. In using soft pencils, also, a portion of the lead is easily rubbed off by the friction of the ruler, 'and this is deposited on the surface of the paper in the form of dust, which tends to give the drawing a smeared and dirty appearance.

Probably the best pencil for general work will be found to be an HH or H. A softer pencil may be used for sketching or for general purposes.' The pencil must be sharpened to a fine point, and 'this may

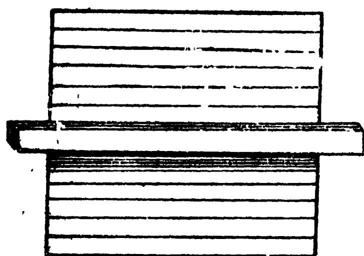
1. In practice the instrument is prepared with *all* its edges straight as shown in Fig. 14 Some distinction is made between a straight-edge and a rule; a straight-edge is a longer piece and is made of metal.

be done by rubbing on a fine sandpaper or a smooth file."[1]

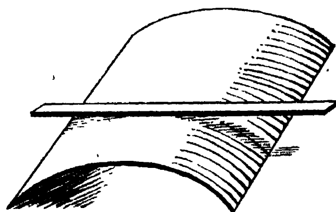
In drawing a straight line through a point always take care that the line you draw really passes through the point.

4. **Flat or Plane Surfaces.** If a surface is pointed out to you, you can generally tell by simply looking at it whether it is flat or curved. The floor of a room, the top of a table are instances of flat surfaces. The surface of a ball or that of a tree are curved surfaces.

To test whether a surface is flat or not apply a straight-edge or ruler to the surface; if the ruler fits the surface *in every position* [Fig. 15 (*a*)] the surface is flat, otherwise it is curved [Fig. 15 (*b*)]. We may *say that a surface is flat if the straight line joining any two points on it lies wholly on the surface.*



(*a*)



(*b*)

Fig. 15

It may however be pointed out that

1. These instructions are taken from '*Machine Construction and Drawing*'—by Frank Castle, (Macmillan & Co. Ltd.)

we may have surfaces which are not flat, yet straight lines can be drawn on them (*in a certain direction*), see Fig. 4 (b).

Flat surfaces are also called **plane surfaces** or simply **planes.**

5. **Solids.** If you consider the objects around you, you will see that each of them occupies some space which is separated from the surrounding space by surfaces. The space occupied by an *uncut* pencil

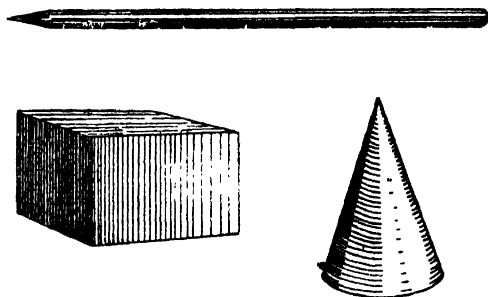


Fig 16—Solids.

for instance, is separated from the surrounding space by a curved surface round the pencil, and two flat surfaces at the ends.

When a body (such as a pencil, or a table) occupies an amount of space bounded by a surface, or surfaces, we call it a **solid.**[1]

1. Not to be confused with '*solid*' as opposed to liquid or gas.

EXERCISE I.

(*a*)

1. Look round the class room and find examples of

(i) surfaces, lines, points, solids ;

(ii) two flat surfaces meeting in a line (is the line straight or curved ?) ;

(iii) two straight lines meeting at a point ;

(iv) three flat surfaces meeting in a point ;

(v) three straight lines meeting in a point ;

(vi) two straight lines which do not meet ;

(vii) two straight lines which do not lie in one plane ;

(viii) two flat surfaces which do not meet.

2. Describe the surface or surfaces bounding the following solids

(i) a pencil cut at one end, (ii) an orange, (iii) an empty glass, (iv) a glass full of water, (v) a closed book, (vi) an open book, (vii) a sheet of thick card-board, (viii) a telegraph post (ix) the face of a man.

3. Are the following *solids* or *surfaces* ?

(i) the leaf of a tree (ii) a piece of orange peel (iii) a sheet of corrugated iron.

4. Give instances of curved surfaces on which straight lines can be drawn.

5. Give an example of a moving point generating a line. [The point of a pencil when moved on a sheet of paper traces out a line].

6. Give an example of a moving line generating a surface (A long piece of a chalk moved sideways on the floor traces out a white surface).

7. How can you move a straight line without tracing a surface ?

8. Give instances of solids (i) bounded entirely by plane surfaces, (ii) bounded entirely by one curved surface, (iii) boun-

ded by two or more curved surfaces, (iv) bounded partly by plane surfaces and partly by curved surfaces.

9. Can a solid be bounded *entirely* by two plane surfaces ? by three plane surfaces ?

10. Give instances of flat surfaces bounded entirely by straight lines [The floor of a room]

11. Give instances of surfaces bounded entirely by one curved line [The top of a round table].

(*b*)

12. Place a dot or point on the table ? How many straight lines can you trace on the table all passing through this point ? With the help of a piece of string, could you lay out a straight line from this point, not lying on the surface of the table ? If so, how many such lines could you lay out ?

13. How many points could you mark on a line ?

Note I. If three or more straight lines pass through the same point they are said to be **concurrent** (Fig. 17).

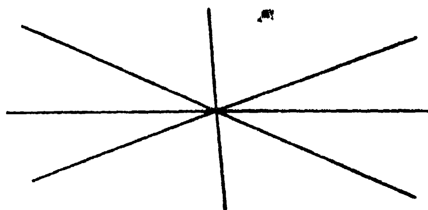


Fig. 17

Note II. If three or more points lie on the same straight line they are said to be **collinear** (Fig. 18).



Fig. 18

Note III. Three or more flat surfaces may pass through the same straight line (Fig. 19).

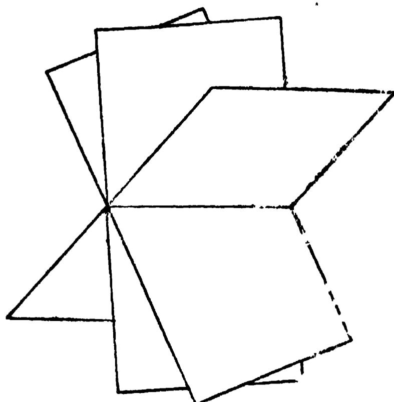


Fig. 19

14. Place three points on paper. How would you test whether they are collinear or not ?

15. Two pegs are driven into the ground wide apart, how would you trace the straight line between them ?

16. Place three points on the paper so that they are not collinear. How many straight lines do you obtain by joining them two by two ?

17. Place four points on a sheet of paper so that no three of them are collinear. Join the points two by two How many straight lines do you obtain ? Do you get any new points from the intersections of these lines two by two taken in all possible ways ?

6. Have true surfaces any thickness, true lines any breadth, and true points any extension ?*

* This mode of presentation is due to the great thinker W. K Clifford—'*The Common Sense of the Exact Sciences*' (Kegan Paul Trench & Co.)

Consider the surface of water in a tank. The space beneath the surface is filled with water, the space above with air. The surface is something which marks off the water from the air. It is the common boundary of air and water, and as such *has no thickness.* It is neither a thin layer of air, nor a thin layer of water.

Thus *a surface is the common boundary between two adjacent portions of space, but itself occupies no room in space. It is not even a thin layer of solid.*

Note the black spot (Fig. 20) made on the surface of paper. It has a size and occupies room on the surface. Thus a surface, *though it occupies no room in space, has a kind of extension of its own, and may be divided into parts.*

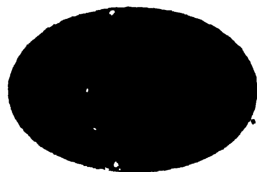


Fig. 20

We also note that the black spot is marked off from the surrounding white surface by a line. It is the common boundary between the two portions of surface (white and black), and as such *has no breadth.* It is neither a thin strip of white nor a thin strip of black.

Thus a line is the common boundary between two adjacent portions of a surface, but itself occupies no room on the surface. It is not even a thin strip of surface.

Now let us suppose the sheet of paper with the black spot on it to be dipped in water, so that a

portion of the black spot is submerged (Fig. 21). The

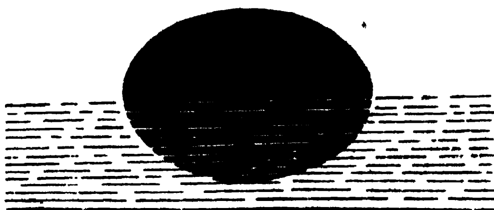


Fig. 21

line (without breadth) which forms the boundary of the black spot is now divided into two parts, one part lying within water, the other out of it. Each of these parts goes a certain way round the spot, and occupies room on the whole line of which it is a part.

Thus a line, though it occupies no room on a surface still less in space, has an extension of its own, and may be divided into parts.

Going back to the two parts into which the line bounding the black spot is divided by the surface of water, we observe that they are marked off from each other by two points, one at each end. These points are not short lengths of either of the parts

Thus a point is a division between two adjacent parts of a line, but itself takes up no room on the line. It is not even a very small length of a line. A point has no extension at all.

We have thus **three kinds of extension** or **room** peculiar respectively to (1) **space** (in which we move obout) (2) **surface** and (3) **line.** *The* **point** *has no extension.*

It must not be supposed that when we speak of a surface as being in space yet not occupying any room in it, of a line being in a surface yet not occupying any room in it, of a point being in a line yet not occupying any room in it, we are speaking of imaginary things. The surface of water which lies where the water and air meet is a real surface which we can see, the line which forms the boundary of the black spot is a real line which we see where the black and the white portions of the surface meet, the points where the two parts of this line meet are real points which we see where the common boundary of the black and white portions of the surface of the paper meet the surface of the water.

The lines that we draw on a sheet of paper or on the blackboard, and the points that we mark with a pencil or chalk are not *true* lines and points. Every such line however thinly drawn has some breadth, and every such point however faintly marked has some extension.

7. **Direction.** When you look *up* or look *down*, look to the *right* or look to the *left*, look *before* or look *behind*, you are looking in different *directions*. 'In what direction do you see the star?' If you were asked this question, you would at once point your finger *straight* to the star and say, 'In this direction'. It really amounts to your indicating the direction by a *straight line* proceeding right *from your eye to the star*. Thus **a direction is pointed out by a straight line.**

But does a straight line, say the straight line AB (Fig. 22), does it indicate



Fig. 22

only one direction? No; it may be understood in two different ways, (i) as pointing *from the point* A *to the point* B or (ii) as pointing *from* B *to* A. This ambiguity may be removed by the use of an *arrowhead.* The straight line **AB** with an arrowhead placed at **B** [Fig. 23 (a)] would indicate the direction *from* **A** *to* **B**, or as we may more simply call it, **the direction AB** (note that the letter **A** is put first). The same straight line with the arrowhead placed at **A** would indicate the direction **BA** [Fig 23 (b)]. These two directions, **AB** and **BA**, are said to be **opposite** to each other.

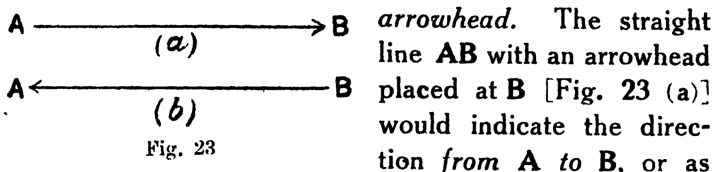


Fig. 23

Motion in a constant direction, change of direction. If you were asked to go from a certain place **A** to another place **B** you might accomplish the journey by a number of different paths as shown in Fig 24.

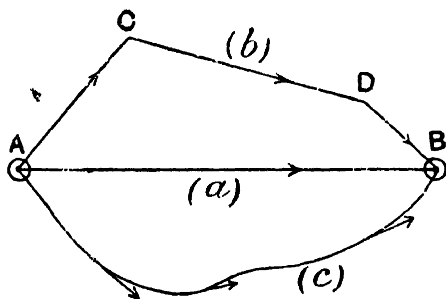


Fig. 24

If you choose the straight path leading from **A** to **B** [Fig. 24 (a)], you would be walking in a *canstant direction all the way*, viz., the direction **AB.**

If you choose the zigzag path **ACDB** [Fig. 24 (b)] you will not be moving in the same direction all the way; from **A** to **C** you will be walking in a constant

direction (the direction AC), but as soon as you come to C you begin to walk in a different direction, (the direction CD) and walk constantly in this direction until you come to the point D; here again for a second time you change your direction of motion, and walk in a new direction (the direction DB) until you reach your destination B.

If you walk along the curved path [marked (c) in Fig. 24] you will be constantly changing your direction of motion, as shown by the arrows.

EXERCISE II.

1. Do you see a cow grazing in the field, always in the same direction?

2. Do you see the sun in the same direction from your room throughout the day?

3 Do you see the morning sun in the same direction from your window, all the days of the year?

4. Do you see the top and the base of a telegraph post in the same direction?

5. How must two objects be situated so that you may see them in the same direction from you?

6 How must two objects be situated so that you may see them in opposite directions from you?

7 Place four points **A, B, C, D** on a sheet of paper. (a) Show the *directions* **AB, CA, CB, BD, AD, CD** by drawing straight lines and placing arrowheads properly. (b) How must the points be situated so that the *directions* **AB, AC, AD** may be one and the same direction? (c) How must the points be situated so that the directions **BC** and **CD** may be opposite to each other? (d) If the directions **BC, CD** are neither the same direction nor opposite directions, can **B, C, D** be collinear?

LENGTHS.

8. Comparison of Lengths :—To compare the lengths of two pencils you would bring them together; if they fitted *exactly* you would say that they were equal in length, if not, one was longer than the other. In the case of two tables, however instead of comparing their lengths directly by placing them in contact with each other, you would rather take a stick, and lying it along the edge of one of the tables mark or cut off a length equal to that of the table. Applying this length to the other table you could know whether the tables were equal in length or not.[1]

If two straight lines AB, CD were traced on a sheet of paper, you could compare their lengths by various devices. One such device is indicated below.

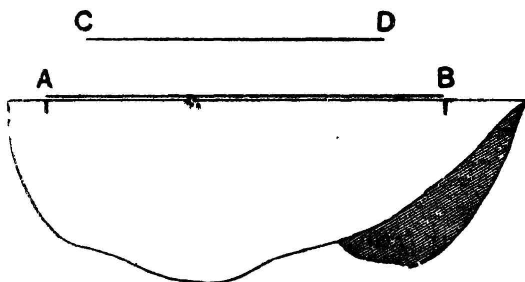


Fig 25

Take a piece of paper of any shape and make a sharp crease in it; bring the crease very close to one of the lines, say, AB, and mark two short cross lines on it, one at each end of the line (Fig. 25). By removing the

1. Could you not compare the lengths of the tables by measuring with a foot rule?

paper thus marked, and applying it to the other line **CD**, you could know whether the lines **AB, CD** were equal in length or not.[1]

With the aid of this same piece of paper, placing it in different positions, *you could trace any number of straight lines each equal in length to* AB.

Ex. 1. Trace any straight line **AB** on a sheet of paper, and mark any point **O** in it. Draw four straight lines from **O** in different directions, each equal in length to **AB.**

Ex. 2. Trace a pair of intersecting straight lines **CD, EF,** and note the point **O** where they intersect. Trace another straight line **AB.** From **A** draw a straight line **AG** equal in length to **CD,** and from **B** a straight line **BH** equal in length to **EF.** Lastly, from **O** draw a straight line equal in length to the straight line **GH.**

Ex 3. Trace a straight line **AB** and mark any point **C** in it. How would you mark another point **D** in the straight line such that **AC** and **BD** may be equal in length. ?

Note. When two straight lines are equal in length, we simply say, '**the lines are equal**' The statement, **AB=CD** means that the straight lines **AB** and **CD** are equal in length.

9. Measurement of lengths. If you were asked to measure the length of a table, you would probably use a foot-rule, and starting from a corner and proceeding along the edge you would measure off first a foot, then another beginning where that ended, and so on, until you reached the other corner. After these 'feet' were marked off you would count their number, and if the number were four you would say 'the length of

1. The lengths could also be compared with a pair of dividers (See foot-note p 22)

the table is 4 feet.' It might however happen that after the foot-rule had been applied four times, a portion was left which was not as much as a foot but less. In order to measure this portion, you would use a smaller unit, the *inch*, and count the number of inches in it; if this number were six, you would say, 'the length of the table is 4 feet 6 inches.' But the portion might not be *exactly* 6 inches, there might be a residue less than an inch. In such a case you would go to measure the residue with a still smaller unit, the *tenth of an inch*, and if there were 7 such units you would say 'the length of the table is 2 feet 6·7 inches.' Even now there is no certainty that the process would end here; for, after the 7 tenths of an inch there might be still another residue which would have to be measured in hundredths of an inch, and so on. Generally, however, nobody wants to know the length of a table more exactly than to the nearest inch. If the residue, after the feet and the inches have been counted, is found to be less than half an inch, we ignore it; and if it is greater than half an inch, we add one more to the number of inches already obtained. This is an *approximate* way of expressing the length. The degree of approximation required in a certain case will depend upon the magnitude of the length, or the distance, to be measured, and also upon the purpose[1] for which the measurement is taken. Nobody would think of

1. If you want to take the measurements of a table for the purpose of buying a table cloth, you will measure the length and the breadth each to the nearest foot *in excess*. But if a carpenter were to take the measurements for the purpose of making another table of equal size, he would measure to the nearest quarter of an inch.

expressing the distance of Benares from Calcutta as so many miles, so many yards, so many feet, so many inches, so many tenths of an inch, and so on. If you look into a Railway Time Table you will see the distance is stated as simply 429 miles (meaning to the *nearest mile*). No man would think of the yards, the feet or the inches that might be left over.[1]

In expressing the length of a pencil, a good approximation would be to the nearest tenth of an inch or to the nearest millimetre. In expressing the breadth of a tape a reasonable approximation would be to the nearest hundreadth of an inch. The length and breadth of a building, the length and the breadth of a tank, the perimeter[2] of a field or playground are generally expressed to the nearest foot. If you go to measure such distances with a foot-rule you will have to apply the rule a large number of times, and in such a process errors are likely to accumulate to a considerable extent. It would be better to use a tape-measure graduated into feet and inches. Such tapes are sold in the market in pieces, 100 or 50 feet long, coiled and put into a leather case (Fig 26).

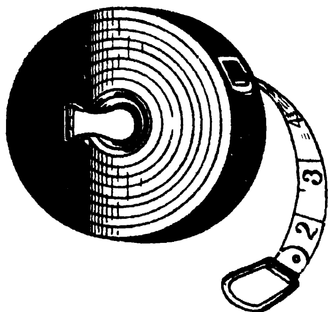


Fig. 26—A tape-measure.

1. Note the absurdity of stating the height of Mount Everest as 29002 feet, as if it were twenty nine thousand and *two* feet *exactly*, and could not be a foot more nor a foot less.

2. The length of the bounding line.

Ex. 1. How would you measure the height of a building ?

Ex. 2 How would you measure the length of a road ?

Ex. 3. How does a cloth-dealer measure out pieces of cloth to his customers ?

Ex. 4. What kind of measure does a tailor use in taking measurements for your coat ?

Ex. 5. You drive two pegs into the ground, how would you measure the distance between them ?

Ex. 6. How would you measure the length of the bounding line of a field, supposing this line to be curved ?

Ex. 7. How would you measure the circumference of a wheel ?

10. **The lengths of lines traced on paper** *are ordinarily expressed to the nearest tenth of an inch or to the nearest millimetre.* This may be done with

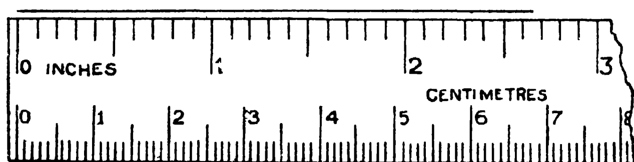


Fig. 27

the aid of a flat ruler graduated into inches and tenths of an inch at one edge, and centimetres and tenths of a centimetre (i. e. millimetres) at the opposite edge. Fig. 27 shows how a measurement is taken. The graduated edge of the scale is brought (by tilting) as near to the line as possible, with one end of the line facing the zero graduation of the scale. The other end may fall *on a division, or between two consecutive divisions.* In the latter case

the line does not contain an exact number of tenths of an inch. In the figure it is observed that the end **B** falls between the 6th and the 7th divisions, and is nearer the 7th division. So we take the length to be 2·7 inches *to the nearest tenth of an inch.* By placing the opposite edge of the scale in contact with the line a measurement to the nearest millimetre might be taken.

In Geometry, '**the distance between two points**'[1] means the length of the straight line joining them, and may be read directly from the ruler by bringing

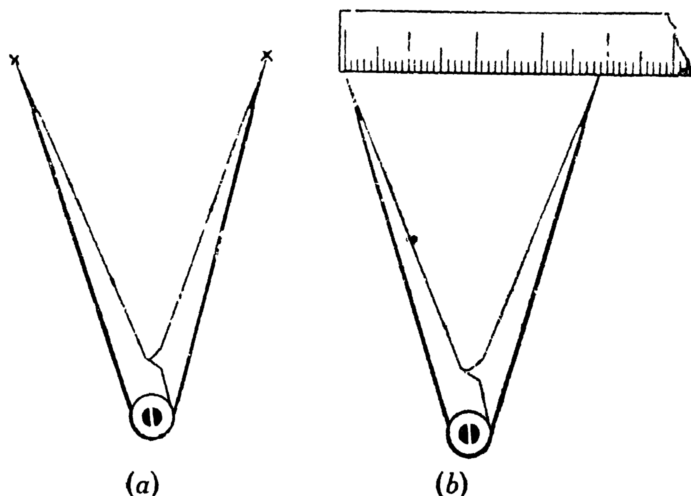


Fig. 28

the graduated edge very close to the points. In

1. When we speak of the distance of a certain place, say the Town Hall from another place, say the Post Office, we generally mean

practice, however, the distance is carried to the scale with the aid of **dividers**[1] (Fig. 28)

EXERCISE III.

All measurements must be taken with great care and precision.

1. Look at the graduations of a ruler and state as correctly as you can (i) the number of centimetres in 4 inches, (ii) the number of inches and tenths of an inch in 9 centimetres. From (i) deduce the number of centimetres in 1 inch, and from (ii) deduce the number of tenths of an inch in 1 centimetre.

2. Measure the distances **AB, BC, DE, PQ, QR, RP,**

(i) in inches to the nearest tenth of an inch ;

(ii) in centimetres to the nearest tenth of a centimetre.

Arrange the results as shown below :—

AB	=	in.	=	cm.
BC	=	in.	=	cm.
DE	=	in.	=	cm.
PQ	=	in.	=	cm.
QR	=	in.	=	cm.
RP	=	in.	=	cm.





Q ×

P ×

R ×

Fig. 29

---

1. An instrument consisting of two legs with pointed ends, capable of turning about a joint called the *head.* [See Fig. 28]. In using dividers the following precautions should be taken, (i) hold them nearly flat so that the points may not injure the scale, (ii) open the legs a little wider than required, and then gradually press them together until the points are at the right distance, (iii) in carrying the distance to the scale hold the dividers by the head only, so that you may not alter the distance between the points, (iv) do not perforate the paper with the points. It may be observed that with

3. Draw four straight lines on the paper. Measure each of them in inches (to the nearest tenth of an inch) and also in centimetres (to the nearest tenth of a centimetre). In each case divide the measure in centimetres by the measure in inches and arrange the results in a tabular form as shown below. Find the average of the results in the last column.

Measure in cm.	Measure in inches	No. of cm's in 1 inch.

4. Measure the lengths **AB, BC, CD** in inches correct to one decimal place. Arrange the results as shown below :—

AB = in.
BC = in.
CD = in.
AB+BC+CD= in
Check by measuring **AD.**



Fig. 30

5. Measure the lengths **AB, BC, CD, DE** in centimetres correct to one decimal place ; and arrange the results as shown below :—

AB = cm.
BC = cm.
CD = cm.
DE = cm.
AB+BC+CD+DE = cm.
Check by measuring **AE.**



Fig. 31

6. Measure the lengths **AC**, **BC** in centimetres correct to one decimal place. Arrange the results as shown below :—

AC	=	cm.
BC	=	cm.
AC−BC	=	cm.

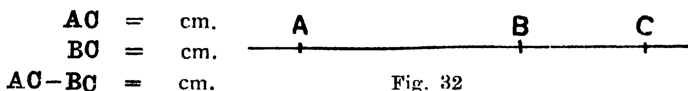


Fig. 32

Check by measuring **AB**.

7. Draw a straight line on the paper sufficiently long and mark off from it parts **AB**, **BC**, **CD** equal respectively to 2·4 cm., 4·7 cm., and 1·3 cm. Add up the lengths and check by measuring **AD**.

8. Draw any curved line on the paper. Mark points **A**, **B**, **C**, in it such that the distances **AB**, **BC** are respectively 3·4 cm and 4·2 cm. Measure the distance **AC** in centimetres correct to one decimal place. Do you find the distance **AC** equal to the sum of the distances **AB**, **BC**, or to their difference ?

9. Mark three points **A**, **B**, **C** on the paper, so that they may not be in one straight line. Measure the lengths **AB**, **BC**, **CA**. (i) Add any two of these and compare the sum with the third length. (ii) Find the difference of any two of these and compare with the third.

10. Draw any straight line and mark two points **A** and **B** in it. Guess the mid-point of **AB** and mark it. Test your guess by comparing the lengths of the two parts with dividers.

11. Draw any straight line and cut off a part **AB** equal to 6·8 centimetres. How would you find and mark the mid-point of **AB** by calculating half its length.

12. Draw any straight line **AB**. How would you bisect* it ?

Note. Could you estimate the length of a line correct to the nearest *hundredth of an inch* or to the nearest *tenth of a millimetre* with the help of the ruler ? With practice and always taking care to bring the edge of the ruler very close to the line, it may be possible to make a fairly correct *guess* of the length to the nearest *hundredth*

* '*To bisect*' means 'to divide into two equal parts.'

of an inch. But there would always remain the doubt as to the accuracy of a result obtained by a guess. You would probably say, 'why not graduate the ruler into hundredths of an inch?' Such a graduation would mean a further subdivision of each tenth of an inch into ten equal parts. Even assuming such a subdivision were correctly performed, each part would be so very minute, and the marks of division would be so very close to each other, that it would be practically impossible to read the measure correctly with bare eyes *

To estimate lengths accurately to the nearest hundredth of an inch, or tenth of a millimetre, a scale of a different type, called the '**Diagonal scale**' (to be explained later, p. 226) must be used.

11. **Lines drawn to scale**—If you look into the map of India in an Atlas, you will see the distance between Calcutta and Benares to be only a few inches on the map, while the actual distance between the two places is over 400 miles. The few inches on the map are therefore to be understood as *standing for* or *representing* a very much larger length. If you look at the bottom of the map you will find a note as shown on the margin. This note indicates the scale on which the map has been drawn, and gives the exact

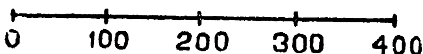


Fig. 32 (a)—English Rule.

length which stands for a distance of 100 miles. If you wish to record the height of Mount Everest, a height of 29000 feet, with a line traced on paper, you cannot think of drawing a line actually 29000 ft. long. You must adopt a scale of representation. Suppose you take a length of 1 inch to stand for 10000 feet, then if you draw a line of length 2·9 inches only, it will completely represent the height of Mount Everest. The line will be said to be drawn to the scale, '1 in = 10000 ft' [Here the sign '=' is to be read

* A lens however might be used.

'*represents*' and not '*equal to*']. The following mountain heights are drawn to a scale of 1 in. to 10000 ft., the lines being traced from top to bottom.

(1) Mount Everest—29000 ft.
(2) Mount Godwen Austin—28250 ft.
(3) Kanchinjanga—28200 ft.
(4) Dhaulagiri—26600 ft.
(5) Mt. St. Elias—19500 ft.
(6) Teneriffe—12000 ft.

By a simple glance at these lines you can form an idea of the relative heights of the mountains.



Fig 33.

Here we are representing a length by means of another length. But other kinds of quantity such as *weights* and *times* may also be represented by lines. Thus if we take $\frac{1}{10}$ inch to stand for the weight of 1 seer, a length of 4 inches will represent a maund.

EXERCISE IV.

1. Draw a straight line to represent the height of a tower 200 feet high, the scale being 1 inch = 100 feet.

2. The distance between Calcutta and Delhi is 900 miles. What is this distance on a map drawn to the scale of $\frac{1}{2}$ in. to 100 miles ?

3. Draw straight lines (from left to right) to represent the lengths of the following rivers on a scale of 1 inch to 500 miles.

The Yenesei—3200 miles The Ganges—1500 miles.
The Amur—3000 miles. The Amazon—4000 miles.
The Brahmaputra—1800 miles The Nile—3500 miles.
The Volga—2200 miles.

4. India extends 2000 miles from North to South, and 2500 miles from East to West. Represent these lengths on a scale of ¼ inch to 100 miles.

5. Represent the melting points[1] of the following metals by straight lines on a scale '½ inch = 100 degrees'.

Copper—1095 degrees centigrade.
Zinc — 418 " "
Silver —1040 " "
Gold —1064 " "
Iron —1500 " "
Aluminium— 660 " "

6. Represent the following weights by straight lines on a scale of 1 inch to 50 pounds :

240 pounds, 112 pounds, 90 pounds, 300 pounds.

7. Represent the following speeds on a scale of ¼ inch to a mile per hour ;

4 miles per hour, 6 miles per hour, 12 miles per hour, 25 miles per hour, 88 yards per minute.

8. Represent the years[2] of the following planets by straight lines on a scale of 1 inch to 100 days.

Mercury—88 days[3] Earth—365 days.
Venus —225 days. Mars—687 days.

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1. *'Melting point'* of a substance means the temperature at which the substance begins to melt.

2. The year of a planet means the period of revolution round the sun.

3. The *terrestrial* day.

ANGLES.

12. **Examples of angles :**—Take a piece of paper and form two creases in it, meeting at a point on the boundary, and forming a corner there. At this corner we have what is called an *angle* (Fig. 34) *bounded by the creases.*

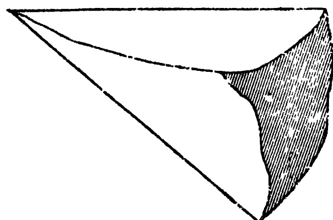


Fig. 34

We have three angles at the three corners of a set-square (Fig. 35).

At the crossing of two straight roads we have four angles (Fig. 36).



Fig 35—Set-square.

Take a pair of dividers, hold one of the legs fixed, and turn the other about the hinge. You have an angle *formed at the head, bounded by the two legs* (Fig. 37).

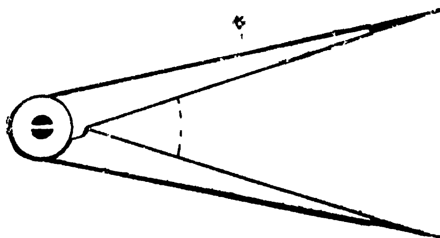


Fig. 37

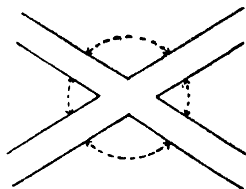


Fig. 36

Thus if two straight lines start from a point, we get a geometrical angle ; the point is called the **vertex,** *and the two lines which bound it are called its* **arms.**

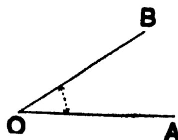


Fig. 38

13. **The size of an angle is independent of the lengths of the arms.** Cut out an angle from a sheet



Fig. 39

of cardboard, and mark a line on it as shown in Fig. 39 (a). If you cut along this line and remove the portion on the right, the angle will remain the same as before, but the arms will be diminished in length Fig. 39 (b). Thus *the size of an angle is independent of the lengths of its arms.*

14. **What is an angle ?** We have seen how an angle is formed, but we do not yet exactly know what it is. Is it the *point* **O** itself (Fig. 40), where the two lines, **OA, OB** meet ? No, the lines which bound it, or what we have called the *arms* of the angle, have got something to do with it. For through the same point

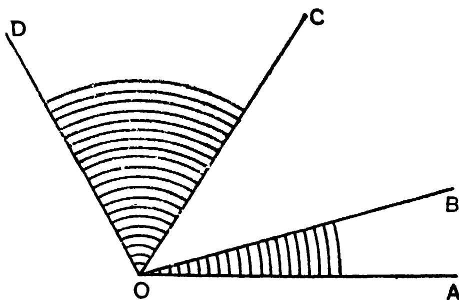


Fig. 40

O we might draw another pair of lines **OC, OD,** (distinct from **OA, OB**) and thereby obtain another angle associated with the same point **O**; but this new angle would be bounded by **OC, OD.** We would thus have two *distinct* angles formed at the same point **O**; one bounded by **OA, OB,** the other by **OC, OD.** Thus an angle whatever it may be, is something which depends upon the straight lines bounding it.

Let us go back to the case of two roads crossing each other. Suppose a man was walking along the road

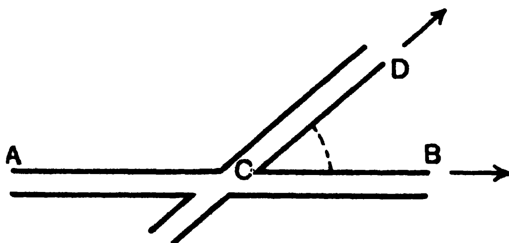


Fig. 41

AB from **A** towards **B** (Fig. 41), but when he came to the crossing **C**, instead of continuing in the direction **CB**, he began to walk in a new direction along the road **CD**. How will you measure this change of direction or to be more precise, the *difference of the two directions* **CB** and **CD**? Here the '*angle*' will come to your help, and you will say that the *angle at* **C** *bounded by* **CB,CD** (Fig. 41) *is a measure of the difference of the directions* **CB, CD.** If the

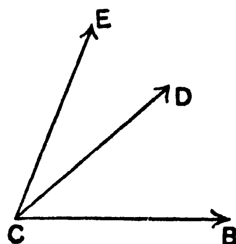


Fig. 42

directions diverged from each other to a greater extent, the angle between them would have been larger in size. Thus, the angle between **CB, CE** is greater than the angle between **CB, CD** (Fig. 42).

The angle may also be looked at from another point of view. Take the case of the dividers (Fig. 37). We may regard the angle between the legs, as being *the measure of the amount of turning required to make one of the legs coincide in direction with the other.*

15. Two ways of regarding an angle : From the illustrations given above, it will be seen that the angle between two straight lines **OA, OB**, may be regarded in two ways.

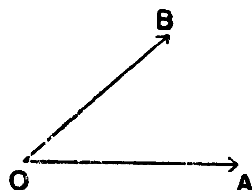


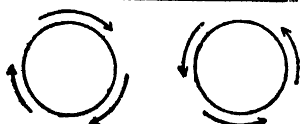
Fig. 43

(1) *It is a measure of the difference of the directions* **OA, OB.**

(2) *It is a measure of the amount of turning* (round the point **O**) *required to bring the direction* **OA** *into coincidence with the direction* **OB**, *or the direction* **OB** *into coincidence with the direction* **OA.**

Note 1. It may be noted that in the second view of the angle, the turning will have to be made in *opposite* ways according as the direction **OA** is brought into coincidence with the direction **OB**, or the direction **OB** into coincidence with the direction **OA**. In the first case **OA** will have to be turned in the *anti-clockwise*[1] way ; in the second case **OB** will have to be turned in the *clockwise* way. But in

1. '*Clockwise way*' means the way in which the hands of a clock move, '*anti-clockwise way*' means the opposite way.


Clockwise. Anti-clockwise.
Fig. 44

both the cases the amount of turning will be the same, and the angle between **OA, OB** is a measure of this amount.

Note 2. Another point to be noticed in this connection is that **OA** may be brought into coincidence with the direction **OB** by an anti-clockwise rotation through the space marked black, or by a clockwise rotation through the white space about **O** (Fig. 45). The latter way will evidently involve a greater amount of rotation than the former. Thus, *there are really two angles* bounded by the same arms **OA, OB**, one being greater in size than the other. To avoid this ambiguity, by the angle between two lines (such as **OA, OB**) we shall always understand the *smaller* one unless we are definitely directed to take the other.

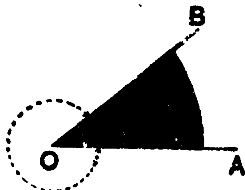


Fig. 45

16. **Equal angles.** Place a set-square flat on the paper and trace two straight lines along the edges, **AB, AC**. You would thus trace an angle on the paper, equal in size to the angle **A** of the set-square. By placing the set-square in different places on the paper you could trace any number of angles all equal to the angle **A** of the set-square, and *hence equal to one another.**

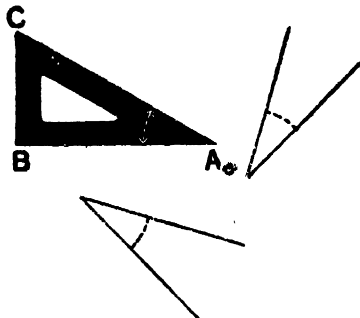


Fig. 46

An angle is given on the paper, how to trace another equal to it? You might cut out the angle

* Things which are equal to the same thing are equal to one another.

and placing it in another part of the paper, copy it with a pencil. But this process would mutilate the paper. Could you not devise a way for *transferring the angle to another part of the paper without taking it up bodily ?*

Take a piece of paper with a sharp crease in it. Place the crease along one of the arms of the given angle. Hold the paper firmly, and make another crease along the other arm of the angle (Fig. 47). You have thus obtained a *portable* paper-**angle** equal to the given angle. [You might remove the folded portions by cutting or tearing through the creases]. With the help of this paper-angle you might trace as many angles as you like, all equal to the given angle.

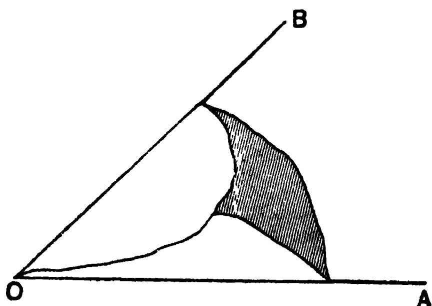


Fig. 47

There is another very *instructive* method* in which the creasing is avoided. Take a piece of paper and mark any point **A** on its edge. Bring this point into coincidence with the vertex of the angle to be copied ; and mark the points **B** and **C** where the arms of the angle meet the edge of the

* This method is taken from Branford's '*Mathematical Education*' (Clarendon Press).

paper* [Fig. 48 (a)]. Now take the piece of paper away and place it in another part of the sheet

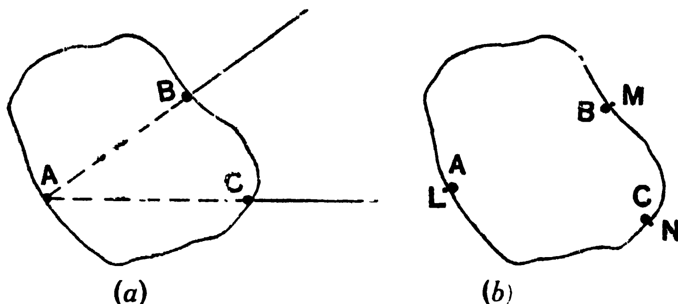


Fig. 48

[Fig. 48 (b)]. Mark the points **L, M, N** on the sheet, corresponding to the three points **A, B, C.** Remove the piece of paper and join **LM** and **LN** ; you will get an angle equal to the original given angle.

Ex. An angle being traced on paper, construct another equal to it,

(i) with the vertex at a given point ;

(ii) with the vertex on a given straight line ;

(iii) with the vertex at a given point on a given straight line ;

(iv) with one of its arms along a given straight line ;

(v) with one of its arms along a given straight line, and the vertex at a given point on that line.

17. Comparison of angles. To compare the size of one angle with that of another, traced on the same sheet of paper.

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* The paper may be of any shape, but the size must be such that the arms of the angle may not be completely hidden from view. Also this paper must not be confounded with the *sheet of paper* on which the angle to be copied has been supposed to be traced.

Let us call the angles **A** and **B**; **B** being bounded by the arms **BP, BQ.**

Form a paper-angle equal to one of the angles, say **A**, and use it to construct an angle equal to **A**, with its vertex at the point **B**, one of its arms along **BP**, and the other arm lying on the same side of **BP** as **BQ**. *If this other arm fell along* **BQ**, *the angle* **A** *would be equal to the angle* **B**. If not, the angles would be unequal.

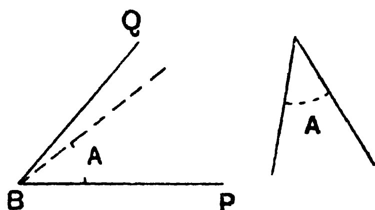
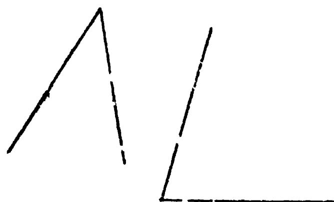


Fig. 49

Suppose the other arm falls *between* **BP** and **BQ** as indicated by the dotted line (Fig. 49), this shows that **B** is the greater angle.*

EXERCISE V.

1. Compare the two angles given on the margin.


Fig. 50

---

* The whole is greater than a part.

A piece of *transparent* paper may also be used in comparing two angles. Place the paper on one of the angles, and make a copy of it. By fitting this copy to the other angle, you could know if the angles were equal or not.

2. Compare the three angles **P**, **Q**, **R**, given on the margin (Fig. 51), and express them in ascending order of magnitude.

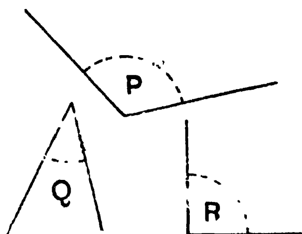


Fig. 51

3. How would you divide a paper-angle (made out of a loose piece of paper) into two halves ?

4. **How would you bisect** (i.e. divide into two equal parts) **an angle traced on paper,** by drawing a straight line through the vertex ? [Form a paper-angle equal to the given angle and use Ex. 3.]

18. Addition of angles. By simply creasing a paper-angle through the vertex, you can divide it into two angles, which together make up the original angle. Cutting through the crease you can separate the two angles. Placing them together again you recover the original angle which is their sum.

Suppose two angles, **A** *and* **B**, *are traced on paper. How would you form an angle which should be as large as* **A** *and* **B** *together?* Form two paper-angles equal to the angles A and B. Placing them together as shown in Fig. 52, you obtain an angle which is as large as A and B put together, or the sum* of A and B. In the same way you may add three or more angles. The sum

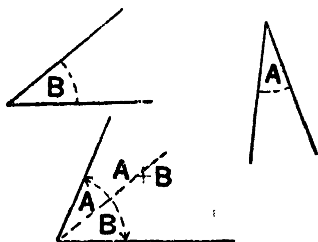


Fig. 52

* If two (or more) angles are added the resulting angle is called their **sum.**

of the three angles **A, B, C** (Fig. 53), is shown below. In Fig. 54 (a), the angles are added in the order **A, B,** and **C**; in (b), they are added in the order

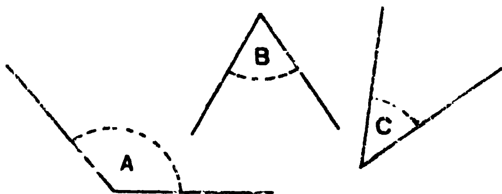


Fig. 53

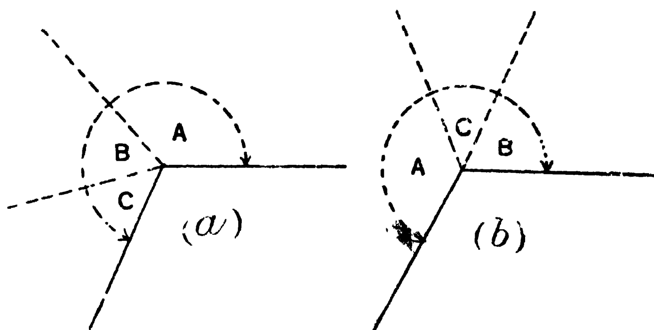


Fig. 54

B, C, A. But the resulting angle (i. e. the sum) is the same in whatever order the angles may be added.

Ex. 1. Trace on the paper an angle which is (i) double a given angle, (ii) 3 times a given angle.

Ex. 2. Mark three points **A, B, C**, on the paper. Draw the straight lines **AB, BC, CA**. Trace an angle which is equal to the sum of the angles formed at the points **A, B, C**.

Ex. 3. Do Ex. 2. with another set of three points.

Ex. 4. Trace two angles on the paper, so that one is greater than the other. Construct an angle equal to their difference.

---

19. **How to name angles.** Suppose we have an angle with the point **O** for vertex, and the lines **OA**, **OB** for arms (Fig. 55). Instead of speaking of this angle as 'the angle between **OA**, **OB**', or 'the angle bounded (or contained or included) by **OA**, **OB**', we can simply call it '*the angle* **AOB**' or '*the angle* **BOA**' (note that the letter **O** is put in the middle). A still shorter way would be to call it '*the angle* **O**', but this would not be possible if there were two or more angles at **O** (Fig. 56).

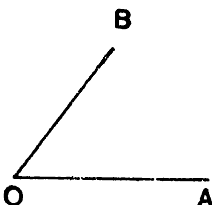


Fig. 55

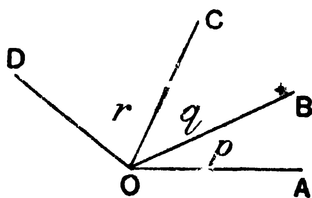


Fig. 56

In referring to an angle in writing, we may introduce further abbreviation by using a sign, '∠', for *angle*. Thus '∠**AOB**' would be read '*angle* **AOB**'.

Sometimes an angle is denoted by an arbitrary letter (usually small) placed between the arms near the vertex. Thus in Fig. 56 *p*, *q*, *r* denote the three angles **AOB**, **BOC**, **COD** respectively.

20. **(Angles at a point—Vertically opposite angles, and Adjacent angles.)**

If two straight lines intersect, four angles are

formed at the point of intersection (see Fig. 57). The angles *p* and *r* are called **vertically opposite angles**; so also *q* and *s* form another pair of vertically opposite angles.

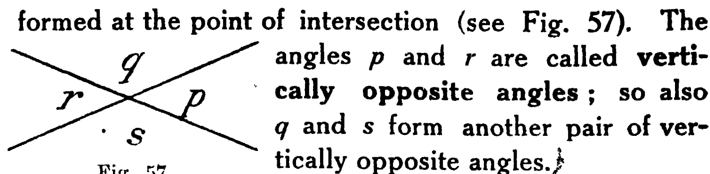


Fig. 57

If two angles have a common vertex and lie on opposite sides of a common arm, they are called **adjacent angles**; X and Y are adjacent angles (Fig. 58).

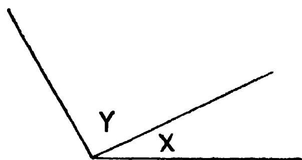


Fig. 58

21. **The Right Angle.** Take a piece of paper and fold it so as to form a crease **AOB** (Fig. 59). Fold it again at **O**, bringing **OA** into coincidence with **OB** in direction (Fig. 60). Now open out the paper and you have four angles at **O** (bounded by the two creases), viz., the angles **AOC, COB, BOD, DOA** (Fig. 61). These *angles are all equal*, for, in the folded condition (Fig. 60), they fitted exactly. Together they make **the angle AOA** (Fig. 61), a peculiar angle with **coincident arms**, which may be regarded

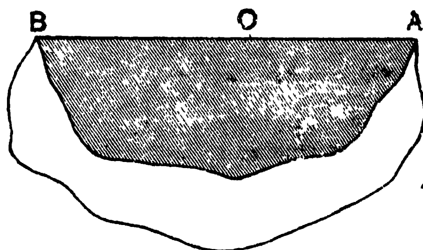


Fig. 59

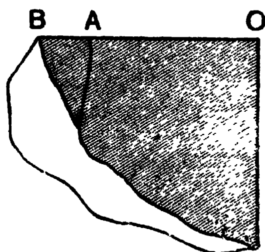


Fig. 60

as measuring the amount of turning (round O as a hinge) required to bring the line OA back to its original position; in other words, **the angle AOA measures a complete turn or rotation.**

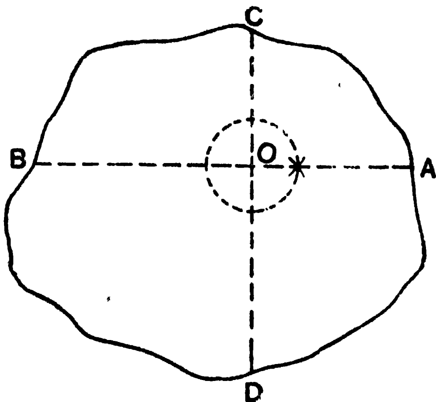


Fig. 61

Each of the four angles (AOC, COB, BOD, DOA) which together make up the angle AOA, is a fourth part of it, and measures a **quarter-turn.** If any other piece of paper were folded in the same way, we would get four angles equal to these (*i. e.*, AOC, COB, BOD, DOA), and each of them also would measure a quarter-turn.

Let us now separate the four angles by cutting through the creases AOB, COD (Fig. 61). With the help of one of these paper-angles you can trace any number of other angles equal to them (Fig. 62). Each of these angles will be called a **Right Angle.**

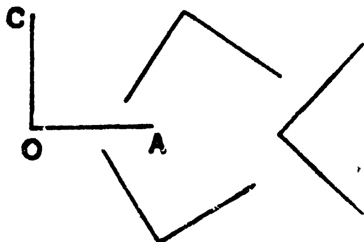


Fig. 62

The angles of a sheet of paper, the angles of a ruler, those at the corners of the top of a table, are all

right angles ; one of the angles of a set-square is a right angle. Give some more examples of right angles.

The angle AOA (Fig. 61) **which measures a complete turn is four times as large as a right angle,** and is consequently **an angle of four right angles.**

If the angle between two lines is a right angle, they are said to be at **right angles.**

22. **Straight angle.** Cutting the piece of paper (Fig. 61) through the crease AOB, we get two portions as shown in Figs. 63, 64. Each of these gives us an angle AOB which is two right angles put together. *The arms* OA, OB *are parts of the same straight line* AOB, *but lie in opposite directions.* Thus *the angle measures the difference between two opposite directions, and its magnitude is two right angles.* It may also be regarded as *measuring a half-turn,* which would bring the direction OA into coincidence with the opposite direction OB.

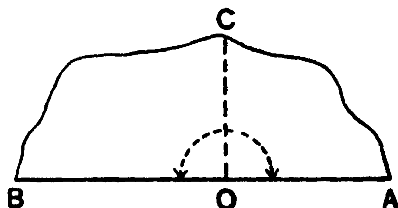


Fig. 63

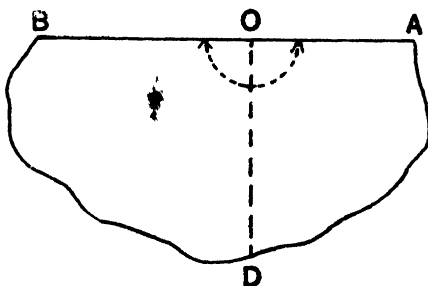


Fig. 64

We shall call this angle a **straight angle*** ; **its magnitude is two right angles, its arms are in one**

* Or a *flat angle* (this name is due to geometer Veronese).

straight line, but in opposite directions, it measures a half-turn.

Note If a straight line **CO** (Fig. 63) meets another straight line **AB** (at **O**), so that the adjacent angles (**AOC** and **BOC**) are equal then each of these angles is a right angle,* and the line **CO** is said to be **perpendicular** to **AB**, or rather the **perpendicular drawn from the point C to the straight line AB.** The point **O** is called the **foot** of the perpendicular. The length **OC** measures what is called the **perpendicular distance,** or simply the **distance of the point C from the straight line AB.**

23. **Angles of any magnitude.** By placing three right angles together we may form an angle **AOD** (Fig. 65) which is three times as large as a right angle.

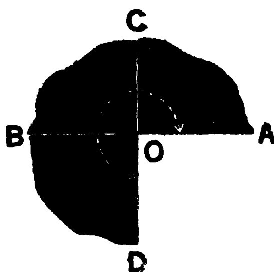


Fig. 65

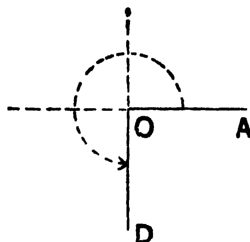


Fig. 66

The same angle could be obtained by turning a line initially coincident with **OA**, through three-quarters of a complete turn (Fig. 66), round the point **O**.

* . This is Euclid's definition of a right angle.

But we might turn the line through any amount, and thereby obtain an angle of any magnitude. Two complete turns would give us an angle of eight right angles (Fig. 67); three complete turns, an angle of 12 right angles, and so on.

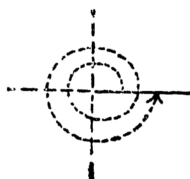


Fig. 67—An angle of eight right angles.

24. **How to trace right angles or draw perpendiculars.** It has already been pointed out (§ 21, Fig. 61) how by cutting through the creases **AOB**, **COD**, you might get as many as four paper-angles, each equal to a right angle, and with the help of one of these you might trace as many right angles as you liked (see Fig. 62). But the drawback of such a paper-angle is that it is not sufficiently stiff. It would be better if you could make one from a stiffer material, say a piece of cardboard. This you could easily do by pasting the paper-angle upon the piece of cardboard and cutting along the arms of the paper-angle (Fig. 68) thus pasted. With the instrument you prepare, you may trace right angles, or by fitting one of its straight edges to any given straight line, you can draw a perpendicular to that line.*

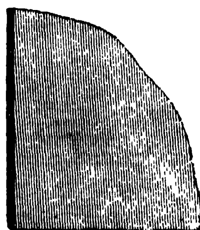


Fig. 68

* Perpendiculars and right angles are generally constructed with the help of set-squares, the construction and use of which will be explained later (see pp. 147, 151).

25. Acute angles and obtuse angles. (If an angle is *smaller than a right angle*, it is called an **acute angle**) (Fig. 69). (If an angle is *greater than a right angle*

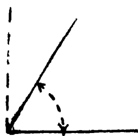


Fig. 69—An acute angle.

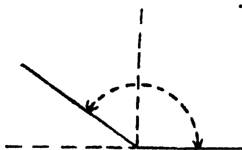


Fig. 70—An obtuse angle.

but less than a straight angle (i.e. 2 right angles), it is called an **obtuse angle**) (Fig. 70).

Note. (If an angle is greater than two right angles, but less than four right angles, it is called a **reflex angle**.)

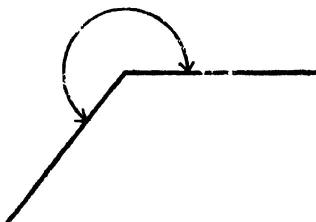


Fig. 71—A reflex angle.

26. Complementary angles and supplementary angles.

(If two angles are such that *their sum is a right angle*, they are said to be **complementary**, or one is said to be the **complement** of the other) (Fig. 72).

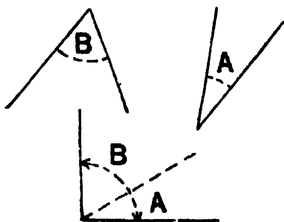


Fig. 72—A+B=1 right angle. Hence A and B are complementary angles.

(If two angles are such that *their sum is a straight angle* (i.e., *two right angles*), they are said to be **supple-**

mentary, or one is said to be the **supplement** of the other (Fig. 73).

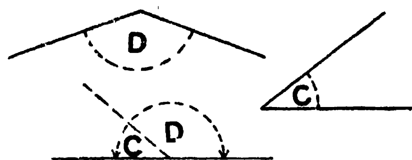


Fig. 73—C + D = 2 rt. angles. Hence C and D are supplementary angles.

Note I. Take a right angle **AOB** (Fig. 74). Through the vertex **O** draw any straight line **OC**, *between* **OA** and **OB**; the angles **AOC, COB** are complementary.

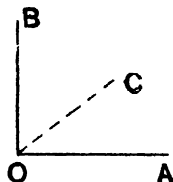


Fig. 74

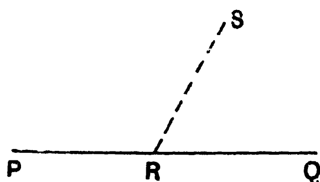


Fig. 75

Note II. Take a straight line **PQ** (Fig. 75), and mark any point **R** in it. Through **R** draw any straight line **RS**; the angles **QRS, SRP** are supplementary.

EXERCISE VI.

(*a*)

1. Draw any straight line **AB** and mark a point **O** in it. From **O** draw a line perpendicular to **AB**.

2. Draw any straight line **AB**, and from its end-points draw two lines each at right angles to **AB**.

3. Draw any straight line **XY** and mark a point **P** outside it. From **P** draw a perpendicular upon **XY**. Measure the distance of **P** from **XY**?

4. Draw two lines **OX**, **OY** at right angles to each other. Mark any point **P** within the angle **XOY**. Draw **PM**, **PN** perpendiculars to **OX** and **OY** respectively; obtain the distances of **P** from **OX** and **OY** by measuring **PM** and **PN**.

5. In Ex. 4, mark two other points **Q** and **R** within the angle **XOY**, and find out their distances from **OX** and **OY**.

6. Take a straight line **AB** of length 3 inches. From **A** draw a straight line at right angles to **AB**, and on this line take a point **C** such that **AC** = 4 inches. Join **BC** and measure it. Do you find **BC** = $\sqrt{3^2+4^2}$ inches?

(*b*)

7. An acute angle being traced on paper, how would you obtain its complement?

8. An obtuse angle being traced on paper, how would you obtain its supplement?

9. An obtuse angle being traced on paper, how would you find its excess over a right angle?

10. An acute angle being traced on paper, how would you obtain its supplement?

11. What is the size of the supplement of a right angle?

12. Can two obtuse angles be complementary?

13. Can two acute angles be supplementary?

14. Can two obtuse angles be supplementary?

15. Can two angles (one acute, the other obtuse) be supplementary?

16. Can two acute angles be complementary?

17. An angle being acute, will its complement be acute or obtuse or right?

18. An angle being acute, will its supplement be acute or obtuse, or right?

19. (An angle being obtuse, will its supplement be acute or obtuse or right ?)

20 (If two acute angles are equal, will their complements be equal ? If two angles are each a complement of a third angle are the two angles equal ?)

21. (If two acute angles are unequal, can their complements be equal ?)

22. (If two angles are equal, will their supplements be equal ?)

If two angles are each a supplement of a third angle, are the two angles equal ?)

23. (If two angles are unequal, can their supplements be equal ?)

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27. How to divide an angle into a number of equal parts.

Take a paper-angle AOB, and crease it through the vertex so that *the arms* OA, OB *come to lie one upon*

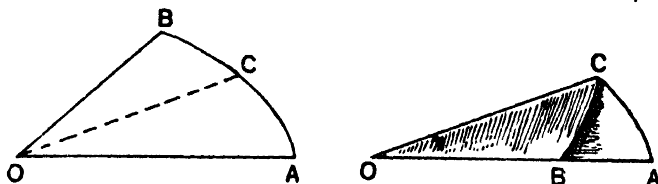


Fig. 76—Bisection of an angle.

the other. Evidently the crease **bisects** the angle, i. e., *divides it into two equal parts* (Fig. 76). Cutting through the crease and taking one of the parts, you may divide it again into two equal parts, as before. Each of the new parts would be a *fourth* part of the original angle. Continuing the process, you might get an *eighth* part, a *sixteenth* part, and so on.

But you would not get a third part. To divide an angle into three equal parts you will have to form *two*

creases through the vertex by folding twice, as shown

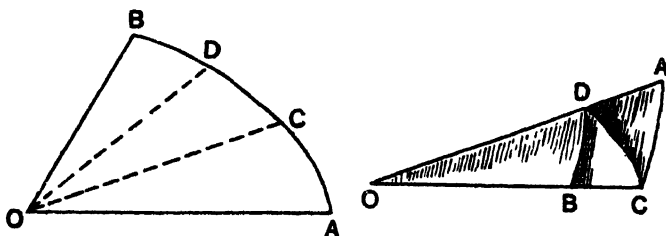


Fig. 77—Trisection of an angle.

in Fig. 77. With a little care you can do it (provided the angle is not small). Cutting through the creases, and taking one of the three parts, you may **trisect** it again as before. Each of the new parts would be a *ninth* part of the original angle. By continuing the process you may obtain 27th. part, an 81st. part, and so on.

By combining the processes of bisection and trisection you may get a *sixth* part, a *twelfth* part, an *eighteenth* part, and so on. But you won't get a fifth part, or a tenth part, or a fifteenth part, etc. To obtain such parts you should learn how to divide an angle into *five* equal parts by forming *four* creases as shown in Fig. 78. But this is by no means a simple thing to do.

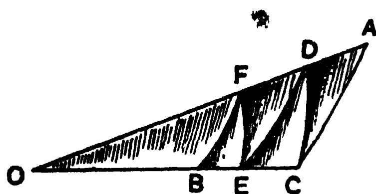


Fig. 78

Though it has been explained in a general way how an angle might be divided into a number of equal parts by *creasing*, in actual practice it is not possible

(specially if the angle be not sufficiently large) to carry on such a process of division beyond bisection or trisection. (See foot-note, p. 57).

Is there no process by which an angle could be accurately divided into a fairly large number of equal parts ? Yes, there is a method, but to understand it, you will have to study the properties of a certain curved line, called the **circle.**[1]

28. **The circle.**—Open a pair of *compasses*[2] to a certain distance, say, an inch ; this you can do by applying the compasses to a graduated ruler (see Fig. 79). Holding by the head and placing the *steel-point* on the

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1. A curved line with which you are quite familiar ; you have it in clockdials, round tables, gramophone discs, wells, filter papers, wheels, coins such as rupees, pices, pies etc.

2. **Compasses,** like dividers, consist of two legs joined at the head (Fig. 79). One of the legs terminates in a fine steel-point, the other is fitted with an arrangement for carrying a pencil or inking pen. The head is provided with a screw adjustment by which the joint may

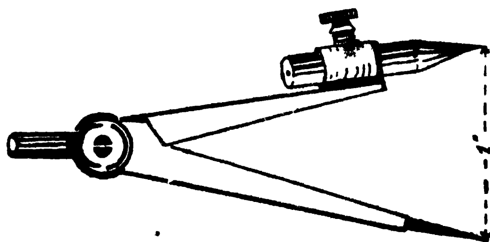


Fig. 79

be tightened or released. This adjustment is necessary, for if the joint is too stiff, it is difficult to open the compasses, and if it is not stiff enough, the distance, to which the compasses may be opened, is liable to alteration while the instrument is carried from one place to another.

The pencil must be sharpened to a fine point. With ordinary compasses such as shown in Fig. 79, you can draw circles up to 9 or 10 inches radius by fitting a full sized pencil.

paper, say at the point O (Fig. 80) rotate* the instrument. The *pencil-point* will trace out a *curved line*; this line is called a circle. From the very manner of tracing it, it is evident that all *the points on it are* at the same distance (*viz.*, one inch, the distance to which the compasses were originally set)

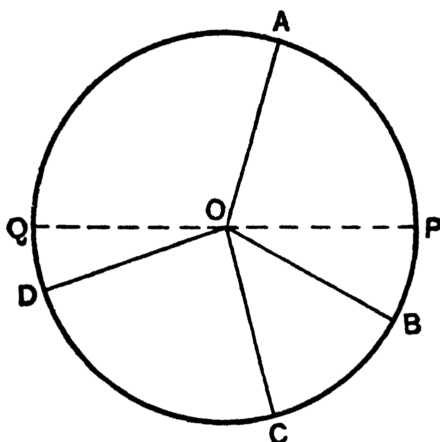


Fig. 80

For drawing larger circles you may use what is called a Beam-Compass which consists of a rod, usually 24 inches long with two sliding pieces having steel points which may be fixed by means of thumb-screws on top (Fig. 81).

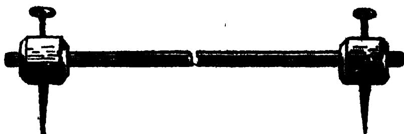


Fig. 81

For blackboard drawing, Blackboard-Compasses (consisting of two rounded wooden legs) are used. The legs are generally 16 inches long (Fig. 82).

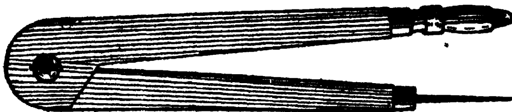


Fig. 82

* In rotating the compasses care must be taken that the steel-point is always kept at the same point O, and the pencil-point always *in contact* with the paper. Care must also be taken that the distance between these points does not alter while rotating the instrument.

from the point **O**. This point **(O)** is called **the centre,** and this distance, the **radius** of the circle.

With the help of compasses you can trace circles with different centres and different radii. In particular you may draw two or more circles of different radii, but with their centres at the same point; such circles are said to be **concentric** (*having the same centre*).

By the word *radius** we may also mean the straight line joining the centre to a point on the circle. Thus **OA, OB, OC, OD** are all radii of the circle (Fig. 80). Evidently *all radii of a circle are equal.*

Let us draw a straight line through the centre **O**, meeting the circle in two points say **P** and **Q** (Fig. 80). Such a straight line (passing through the centre and terminated both ways by the circle) is called a **diameter.** Evidently a diameter is twice as long as a radius, and its mid-point is the centre of the circle; thus all diameters are equal and concurrent (see Note 1, p. 9), and bisect one another.

29. Sometimes the word '*circle*' is used to denote, *not the line*, but the *surface* bounded by the line. In such a case the line itself is called the **circumference** of the circle.

Any portion of the circumference, such as **AB**, (Fig. 83) is called an **arc** of the circle. The *arc* **AB**, which is a curved line must be **distinguished** from the *straight line* **AB** which joins the points **A** and **B**.

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* The Greeks had no word corresponding to *radius*; if they had to express it, they said, "(straight line) drawn from the centre." [Heath—*The Thirteen Books of Euclid's Elements*, Vol. I. p. 199.]

When it is necessary to make a distinction between the two, we may use the notation, $\overparen{AB}$, (note the sign '$\smile$') for *arc* **AB**, and simply **AB** (with no sign over it) for the *straight line* **AB**. This *straight line* **AB** *which joins two points on the circumference* is called a **chord** of the circle.

The portion of surface bounded by an arc of a circle and the two radii joining the centre to the extremities of the arc, is called a **sector** of the circle. Thus **OCD** is a sector of the circle (Fig. 83).

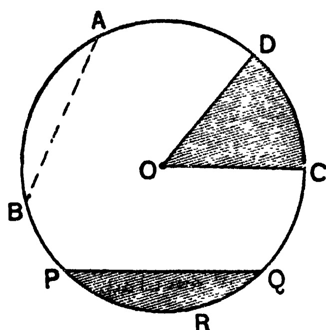


Fig. 83

The portion of surface bounded by an arc of a circle, and the chord joining the extremities of the arc is called a **segment** of the circle. Thus **PRQ** (bounded by the arc **PRQ** and the chord **PQ**) is a segment of the circle (Fig. 83).

EXERCISE VII.

1. Describe in your own words what you mean by 'a circle', 'centre' and 'radius' of a circle.

2. Draw two circles each of diameter 3 inches.

3. Draw a circle of diameter 6 centimetres, and take any point on its circumference; with this point as centre draw another circle of radius 2·5 centimetres.

4. How would you trace on the ground a circle of 100 feet radius?

5. How does a gardener mark out a circular bed?

6. How does a carpenter mark out a circular top for a round table?

7. Take any point **O** and mark 6 points all at a distance of 2·5 inches from **O**; on what curve do these points lie?

8. Take any point **O** and draw 4 straight lines from it. Explain how by drawing a circle you can cut off lengths **OA**, **OB**, **OC**, **OD**, from these lines, each equal to 2 inches.

9. Draw three *concentric* circles of radii 1 inch, 1·5 inches, and 2 inches respectively.

10. Draw a straight line **AB**, and trace a circle with **AB** as diameter.

11. Draw a circle with radius 2·5 inches, and take any point **A** on the circumference From **A** draw chords 1·7 inches, 2·8 inches, 3·5 inches, and 4·2 inches long. What is the longest chord that you can draw through **A**?

12. Take a straight line **AB** of length 5 centimetres. With centre **A** and radius 3·5 centimetres describe a circle; with centre **B** and the same radius describe another circle. Mark the two points **C** and **D** where the circles intersect. Join **CD** and mark the point **E** where **CD** cuts **AB**. Measure the distances **AE**, **BE**; do you find them equal? Measure **CE**, **DE**, do you find them equal?

13. Take a straight line **AB** of length 4·5 inches. With **A** as centre describe a circle of radius 1·5 inches, cutting the line **AB** in the point **C**. *Calculate* the length of **BC** and verify by measurement.

14. In Ex. 13, draw another circle with the point **B** as centre. and radius 2 inches, cutting **AB** in **D**. Calculate the length of **CD** Verify by measurement.

15. In Ex. 13, draw another circle with the point **B** as centre and radius 3·7 inches, cutting **AB** in **E**. Calculate the length of **CE**. Verify by measurement.

16. Draw two circles of radii 1 inch and 1·5 inches respectively, with their centres, (i) 3 inches apart, (ii) 2·5 inches apart, (iii) 2 inches apart. Do the circles cut each other in all cases? In case (iii) measure the length of the common chord, and find by measurement whether this chord is bisected by the line joining the centres.

17. Take an angle **AOB** (Fig. 84); with centre **O**, and radius of any length, draw a circle, cutting off from the arms **OA**, **OB** two *equal* lengths **OC**, **OD**. With centres **C** and **D**, and the *same* radius (which may be of any *suitable*[1] length), draw two arcs of circles, cutting at **E**. Join **OE**. Compare the two angles **AOE** and **EOB** by making a paper-angle equal to one of them (§ 17). Do you find them equal?

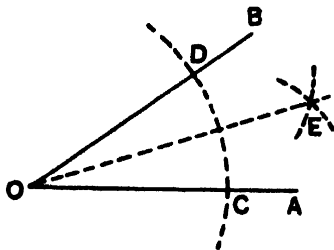


Fig. 84

30. The circle continued.—Mark a point **O** on a piece of paper and with this point as centre trace a circle of any radius. Cut the circle out carefully and make a fold through the centre **O**. The crease **AOB** thus formed (Fig. 85) is a diameter of the circle.[2] The two parts into which the circle is divided by the crease will be found to fit exactly in the folded stage (Fig. 85), and are therefore equal. Thus *any diameter divides the circle into two equal parts*. Each of these parts is called a **semi-circle**.

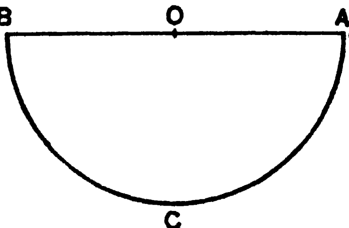


Fig. 85

1. The radius must be of such length that the circles may *intersect*.
2. As it passes through the centre.

Ex. Take a piece of filter paper. How could you find its centre ?

Now make another fold through **O** (see Fig. 85) so that **OA** may fall along **OB**, then the point **A** will fall on the point **B**, because **OA, OB**, being radii, are equal. After this double folding the paper will appear as shown in Fig. 86. The four parts into which the circle is divided by the two creases will be found to fit exactly in the folded stage and are therefore equal ; each of them is a *fourth part of the circle*, and is called a quadrant. Open out the paper, and indicate the creases by dotted lines (Fig. 87). These creases are diameters of the circle as they both pass through **O**, (the centre) and the four angles **AOC, COB, BOD, DOA** are right angles (see § 21). The *arcs* **AC, CB, BD, DA**, are all equal and each of them subtends* a right angle at the centre, i.e., a fourth part of the whole angle of 4 right-angles about the point **O.**

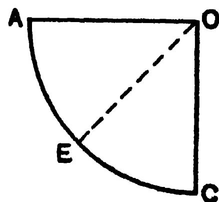


Fig. 86

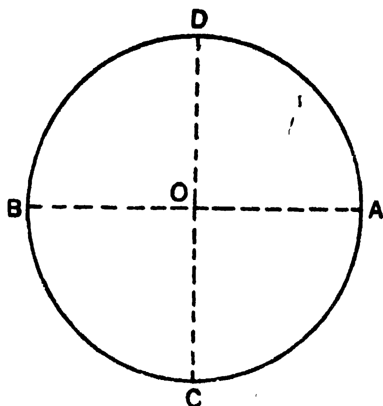


Fig. 87

* If the extremities **A, B** of a line are joined to any point **O** the angle **AOB**, thus formed, is said to be **the angle which the line AB subtends at the point O.**

If you fold again through **O** (see Fig. 86) so that **OC** may fall along **OA** (and consequently **A** on **C**), the circle would be divided into eight equal parts, and each of the angles **AOC, COB, BOD, DOA** (Fig. 87) would be divided into two equal parts. When opened out, the circle will appear as shown in Fig. 88.

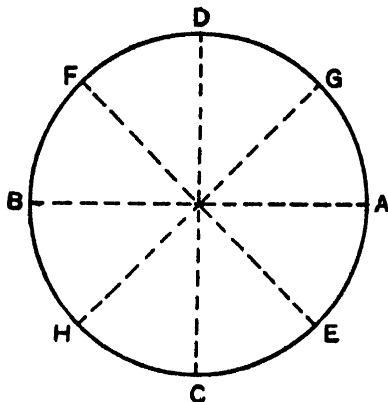


Fig. 88

The whole circumference is now divided into eight equal arcs subtending equal angles at the centre, each of which is half a right-angle, or an eighth part of the whole angle of four right-angles lying about the centre.

By continuing this process of folding you may divide the whole circumference into 16, 32, 64 equal parts, or even a greater number of parts. And the angles subtended at the centre by these equal arcs would also be equal to one another. But in actual practice the process cannot be carried on correctly beyond a certain stage.*

31. From the above considerations you will come to realise that *if the circumference were divided into*

* As the number of folds increases, the paper gains in thickness, and the creases cease to be sharp and well-defined ; moreover, it becomes more and more difficult to make new creases so as to pass *exactly* through the centre O.

a number of equal arcs, all these arcs would subtend equal angles at the centre, and the total of all these angles would be four right-angles.

Indeed we may go a step further and say that *if any two arcs, say* **AB** and **CD**, *are equal, the angles* **AOB** *and* **COD** *will also be equal* (Fig. 89); for the sectors **AOB** and **COD** are so very alike in shape and size, that they may be regarded as one and the same sector, only placed in different positions. If one of the sectors were cut out and placed upon the other, they would fit exactly.

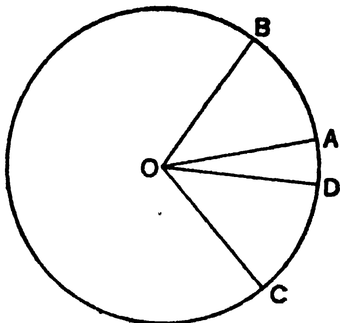


Fig. 89

Ex. 1. It the whole circumference were divided into ten equal arcs, what fraction of a right angle would be the angle subtended by each of these arcs at the centre ?

Ex. 2. If an arc **AB** of a circle of centre **O** be a third part of the whole circumference, how many times a right angle would be the angle **AOB** ?

Ex. 3. If an arc **AB** of a circle of centre **O** be a sixteenth part of the whole circumference, what fraction of a right angle would be angle **AOB** ?

Ex. 4. If the arc **AB** of a circle of centre **O** be twice as long as another arc **PQ** of the same circle, find how many times angle **POQ** is angle **AOB**.

Ex 5. If an arc **OD** of a circle of centre **O** be a sixth part of another arc **AB** of the same circle, what fraction of the angle **AOB** would be angle **OOD** ?

Ex. 6 If two arcs **AB, OD** of a circle of centre **O** be respecttively 3 times, and 4 times a third arc **EF** of the same circle, what fraction of the angle **OOD** would be the angle **AOB** ?

32. Measurement of angles, the Degree. We have seen (Fig. 61) that the angle AOA (with coincident arms) is four times as large as a right angle; we may express its magnitude as 4 *right angles.* A straight angle (§ 22) is twice as big as a right angle, and its magnitude may be expressed as 2 *right angles.* If a right angle were divided into two equal parts, the size of each part would be expressed as ½ *right angle.* Here we are measuring the angles by comparing them with a standard angle, *the right angle.* In other words, the right angle has been adopted as the unit of measurement. To express the size of a given angle in right angles you will have to determine how many times (or what fraction of) a right angle, the given angle is. If an angle is not an *exact* number of right angles, a residue will be left less than a right angle. If you wish to express this residue in right angles, you will have to find out what fraction it is of a right angle. But as yet, you have no means of ascertaining this fraction.

To get over the difficulty you will use a smaller unit. This unit is generally taken to be *the nineti-eth part of a right angle, or 360th part of an angle of four right angles*, and is called a **degree.***

* This is the English unit. The French however take a hundredth part of a right angle as their unit, and call it a *grade.*

Now the question is, 'How to divide an angle of four right angles into 360 equal parts.' We have seen (§ 31) that if the circumference of a circle is divided into a number of equal arcs, all these arcs will subtend equal angles at the centre, and the total of these angles would be an angle of four right angles. Thus, if we *could* actually divide the circumference into 360 equal parts,[1] by drawing straight lines from the centre to the points of division, we would obtain 360 angles each of which would be a 360th part of four right angles i. e. a **degree.** Now this division of the circumference into 360 equal parts (or even a greater number of parts, if required) can be accurately done with the help of a machine called the Circular Dividing Machine.[2] On the margin is given a figure (Fig. 90) showing the division of the circumference into 36 equal arcs.[3] The points of division being joined to the

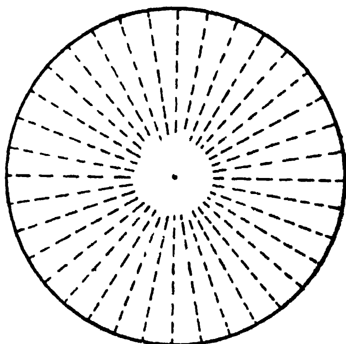


Fig. 90

1. This division of the circumference into 360 equal parts comes from remote antiquity. The ancient Indians as well as the Greeks divided the circumference into 360 parts.

2 It is a mechanical contrivance by which a circular metal disc is made to rotate uniformly round its centre, while a sharp instrument remaining in a fixed place near the edge of the disc and not rotating with the disc makes fine scratches on the rim of the disc at regular intervals. Thus if the disc makes a complete turn in 6 minutes, and a scratch is made every second, the circumference of the disc will be divided into 360 equal arcs by the scratches. A circular Dividing Machine is to be found in the Bengal Engineering College, Shibpur.

3. The rim of a clockdial is divided into 60 equal parts by the divisions indicating minutes.

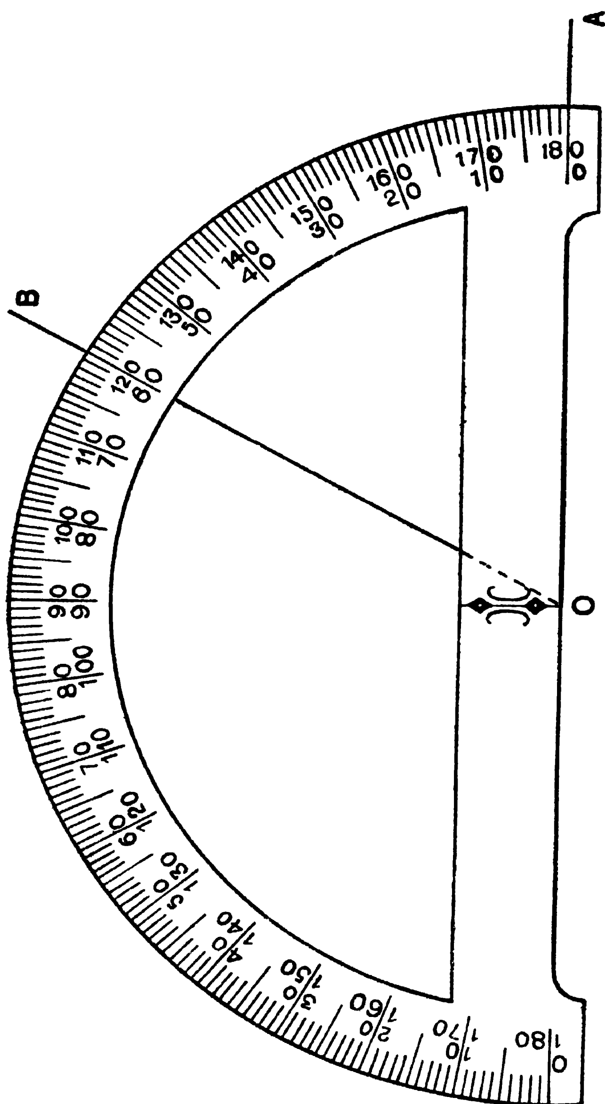


Fig. 91—Semi-circular Protractor.

centre by straight lines, 36 equal angles are obtained, each of which is 10 times a degree, i. e., an angle of 10 degrees. From this figure you get an idea of the magnitude of an angle of 10 degrees. A degree would be a tenth part of it, and you can easily understand how very small an angle the degree is. A right angle is equal to 90 degrees ; a straight angle is equal to 180°, and the angle that measures a complete turn i. e., an angle of four right angles is equivalent to 360 degrees. Two angles are complementary if their sum is 90 degrees, and supplementary if their sum is 180 degrees (§ 26).

33. **The Protractor.** If you take a semi-circle and divide the half-circumference into 180 equal parts, and join the divisions to the centre, you will get 180 angles each equal to a degree. An instrument can actually be constructed on this principle for measuring angles in degrees. Such an instrument is called the **protractor,*** and is usually made of brass or steel plate.

In order to measure an angle **AOB**, place the centre (indicated by a notch) of the protractor at the vertex **O** of the angle (Fig. 91), and the base (the straight edge) of the protractor along one of the arms, **OA**. The number

* Or more particularly a semi-circular protractor.

There are two sets of numbers (as shown in Fig. 91) on the round part of the protractor, the inner set being read when the angle is measured from right to left, the outer set, when it is measured from left to right. A little practice will enable the student to decide which set of numbers to read in a particular case.

It may be noted that the scratches indicating the graduations would all pass through the centre of the protractor if produced straight on.

of degrees in the angle is read off by observing where the other arm, OB, of the angle crosses the round part of the protractor (see Fig. 91). As you see, with a semicircular protractor you can *directly* measure all angles not exceeding two right angles. Could you not use it to measure an angle exceeding two right angles but less than 4 right angles i. e., a reflex angle (see Note, § 25) ?

The protractor may also be used to set out an angle of a given number of degrees (say 40 degrees). For this purpose place the protractor flat on the paper and holding it firmly, trace a straight line along the base of the protractor, and mark two points on the paper, one (O) at the centre of the protractor, and the other (C) at the division indicating 40 degrees (Fig. 92).

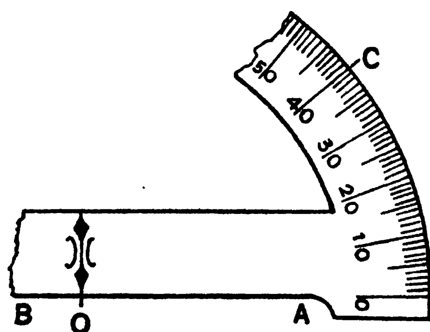
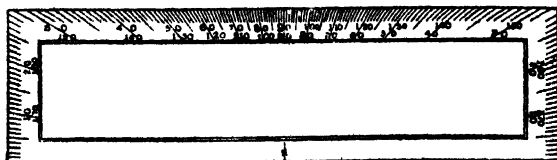


Fig. 92

There is another type of protractor very much in use, called the *rectangular protractor as shown in* Fig. 93.


Fig. 93

Remove the protractor and join **OC**. The angle **AOC** is of the required magnitude (40 degrees).

34. **Minutes and Seconds.** It may happen that the arm **OB** of the angle to be measured (Fig. 91) does not cross the protractor exactly at a graduation but passes between two consecutive divisions; in such a case the angle will not contain an exact number of degrees; a residue, less than a degree, will be left. This residue will be a very small angle in itself, and you may express the magnitude of the original angle to the *nearest degree* by ignoring the residue if it happens to be less than half a degree, or taking one degree more, if you find the residue to be greater than half a degree. But if it is your object to *measure* this residue, you will have to use a unit smaller than a degree. Such a unit is usually taken to be a sixtieth part of a degree and is called a **minute.** If you wish to measure a fraction of a minute, you will have to use a still smaller unit, called the **second** which is again a sixtieth part of a minute.* Thus we have a complete set of units for measuring angles.

1 Right angle = 90 degrees.
1 Degree = 60 minutes.
1 Minute = 60 seconds.

* The degree is a good approximation for ordinary purposes. With a large-sized protractor it is possible to make graduations indicating a sixth or even a twelfth of a degree. Such an instrument would measure an angle to the nearest 5-minutes. In Astronomy, where a much greater approximation is required in the measurement of angles, an instrument called the **sextant**, is used. With a sextant it is possible to measure angles to the nearest 5-seconds.

This is the English system. The French, however, use a different scale of units :

1 (Right angle = 100 grades.
1 Grade = 100 minutes.
1 Minute = 100 seconds)

The French minutes and seconds must not be confused with the English minutes and seconds. Also, minutes and seconds of *angular measure* must not be confused with minutes and second of *time.*

An angle of d degrees is denoted by d°, an angle of m minutes by m', and an angle of n seconds by n''; thus the angle $40^\circ\, 25'\, 49''$ means an angle which contains 40 degree + 25 minutes + 49 seconds. The abbreviation for a right angle is 'rt. ∠.'

EXERCISE VIII.

(*a*)

1. Express in degrees the angle which an eighth part of the circumference, subtends at the centre of a circle.

2. If a right angle is divided into 4 equal parts, what will be the value of each part (i) in degrees, (ii) in degrees and minutes ?

3. If a right angle is divided into 13 equal parts what will be the value of each part in degrees, minutes and seconds (to the nearest second) ?

4. If the circumference of a circle is divided into 12 equal arcs, what will be the measure (in degrees) of the angle subtended by each of these arcs at the centre ?

5. Express the following angles in degrees :—(i) $\frac{2}{3}$ of a right angle, (ii) $1\frac{1}{3}$ right angle, (iii) 2·35 right angles.

6. What fractions of a right angle are the following angles ?
(i) 20° (ii) 35° (iii) 112° (iv) $67^\circ\ 30'$ (v) $10^\circ\ 40'\ 50''$

7. If two angles are respectively 50° 27′ 49″ and 107° 50′ 28″, what will be the magnitude of the angle obtained by adding these two angles ?

8. What must be the magnitude of the angle which being added to an angle of 72° 15′ will make a right angle ?

9. What fractions of a right angle are the angles between the hands of a clock at the following times (i) 1 P. M. (ii) 1 A. M. (iii) 3 P. M. (iv) 7 A. M. (v) noon ? Express the angles also in degrees.

10. Through what angles does the minute hand of a clock turn in (i) one hour, (ii) half an hour, (iii) 10 minutes ? Give the answers in degrees.

11. Through what angle does the hour-hand of a clock turn in (i) one hour (ii) 5 hours, (iii) 12 hours, (iv) 18 hours, (v) 24 hours, (vi) 4 hours and 48 minutes ? Express the angle in degrees in each case.

12. Express in degrees the angle between the two hands of a clock at 15 minutes past seven.

(b)

(On measuring angles with a protractor.)

13. Trace three angles with sufficiently long arms and measure each of them with a protractor.

14. Mark three points **A**, **B**, **C**, on the paper, and join **AB**, **BC**, **CA**. Measure the angles at **A**, **B**, **C** with a protractor (producing the arms if necessary).

15. Measure the two angles **AOB**, **BOC**, given below ; add the results and check by measuring the whole angle **AOC**.

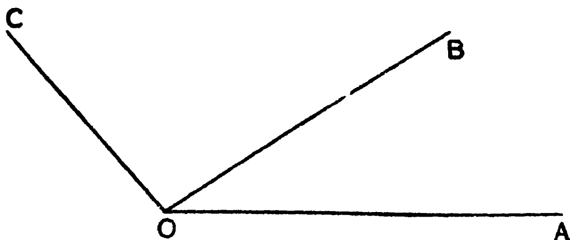


Fig. 94

16. Measure the angles **POQ**, **QOR**, **ROS**; add the results and check by measuring the whole angle **POS**.

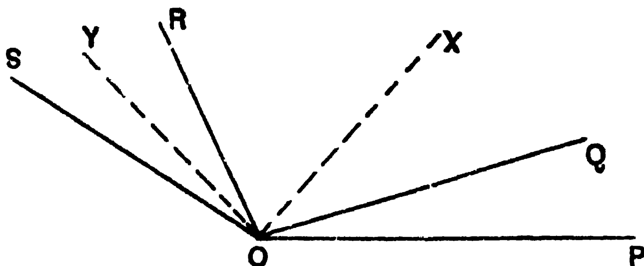


Fig. 95

17. In Fig. 95, measure the angles **POX**, **POY**; find their difference, and check by measuring the angle **XOY**.

18. In Fig. 95, measure the angles **POQ**, **QOR**, **ROS**, and add these results. Also measure the angles **POX**, **XOY**, **YOS**, and add the results. Compare the two sums. Do you expect the two sums to be equal?

19. Draw a reflex angle (see Note, § 25): how would you measure it with the aid of a semi-circular protractor?

20. Mark three points **A**, **B**, **C**; join **AB**, **BC**, **CA**, and measure the angles at **A**, **B**, **C**; add the results.

21. Proceed exactly as in Ex. 20, with another set of three points **A′**, **B′**, **C′**.

22. Do you find the sum of the angles **A**, **B**, **C** in Ex. 20, equal (or nearly equal) to the sum of the angles **A′**, **B′**, **C′** in Ex. 21? What is the sum? Is it equal (or nearly equal) to 180 degrees?

23. Draw any straight line **AB**, and mark a point **O** in it. From **O** draw another straight line **OC**. Measure the angles **AOC**, **COB** and add the results. Do you expect the sum to be 180°? If so, why? Do you actually find the sum (as obtained by adding the measures) to be exactly equal to 180°? If not, how will you account for the discrepancy?

24. Draw two straight lines **AOB, COD** (as shown in **Fig. 96**). Measure the angles **AOC, COB, BOD, DOA**, and add the

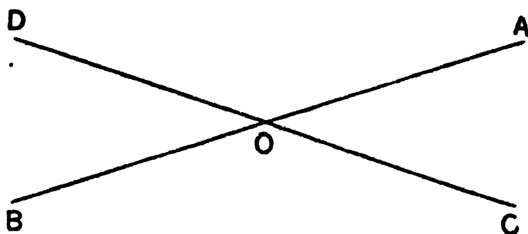


Fig. 96

results. Do you expect the sum to be 360°? If so, why? Do you actually find the sum (as obtained by adding the measures) to be *exactly* 360°? If not, how are you to account for the discrepancy?

25. In Ex. 24, do you find the measure of the angle **AOC** to be equal (or *nearly* equal) to that of the angle **BOD**? Do you expect the angles to be exactly equal?

26. In Ex. 24, do you find the measure of the angle **BOC** to be equal (or *nearly* equal) to that of the angle **AOD**? Do you expect the angles to be exactly equal?

(c)

(Complementary angles, and supplementary angles.)

27. Calculate the magnitudes of the complements of the following angles (i) 50° (ii) 72° (iii) 25° 48′ (iv) 61° 35′ 40″.

28. Calculate the magnitudes of the supplements of the following angles. (i) 52° (ii) 100″ (iii) 120° (iv) 150° (v) 45° (vi) 22° 18′ 30″ (vii) 132° 8′ 54″.

(d)

(On tracing of angles of given magnitudes.)

29. Draw very carefully with a protractor the following angles. (i) 30° (ii) 60° (iii) 145° (iv) 67° (v) 85° (vi) 120° (vii) 105°. State which of these angles are acute, and which obtuse.

30. Set out an angle **AOB** of magnitude 55°. Produce **AO** to **C**, and calculate the angle **COB**. Verify by measurement.

31. Draw a line **AB**, 7 cm. long, and make angles **BAC** and **ABC** of magnitudes 70° and 48° respectively. Measure the angle **ACB** and also the lengths **AC**, **CB**.

32. Make an angle of 70° at any point **O**, and from the arms cut off lengths **OP**, **OQ** equal respectively to 2 inches and 3 inches. Join **PQ**. Measure **PQ**, and the angles **OPQ** and **OQP**.

33. Mark a point **O** on the paper. From **O** draw straight lines **OA**, **OB**, **OC**, **OD**, so that ∠**AOB**=30°, ∠**BOC**=48°, and ∠**COD**=102°. Do you find the extreme arms **OA**, **OD** in one straight line?

34. Make an angle of 150°, and bisect it.

[The making of the angle and its bisection can be done simultaneously as follows. The half-angle is found by calculation to be 75°. Placing the protractor on the paper, draw a straight line along its base and mark three points—one at the centre of the protractor, another at the 150° graduation, and the third at the 75° graduation. Join each of the last two points to the first.]

35. Make an angle of 120° and trisect it.

36. Draw any angle **AOB** (acute or obtuse). How would you (i) bisect it, (ii) trisect it, with the aid of a protractor?

37. Make two angles **BAC** and **BAD** (on opposite sides of **AB**) of magnitudes 60° and 40° respectively. Calculate the angle between the bisectors of the two angles. Verify by actually drawing the bisectors and measuring the angle between them.

38. Do Ex. 37, taking the ∠s **BAC, BAD**, on the same side of **AB**.

35. The length of the circumference of a circle. How would you measure the circumference of a wheel? Make a chalk mark on the rim, place the wheel on a level piece of ground, with the chalk mark on the ground, and make it roll in a straight line until the chalk mark comes again to the ground. The wheel has made one complete revolution. If you measure the distance between the two positions of the chalk mark on the ground, you will get the length of the circumference of the wheel. In the same way you may determine the circumference of a rupee-coin by rolling it once round on a piece of white clean blotting paper.

Such a method won't do if you are to measure the length of the circumference of a circle traced on paper. In such a case you may place (or paste) the paper on a piece of card-board and stick pins all along the circumference. Round these pins[1] put a thread and cut off from the thread a portion which is just sufficient to go round the circumference once. The length of this piece of thread will give you the length of the circumference.

Ex. 1. A wheel makes 600 revolutions in going over a mile. Calculate the length of the circumference.

Ex. 2. Draw a circle of radius 1·5 inches, and use a thread to measure the circumference. Express the length to the nearest tenth of an inch.

---

1. You must take a large number of pins and stick them very close together along the circumference.

Ex. 3. Trace three circles of diameters 5 inches, 8 inches and 10 inches respectively. With the help of a thread measure the circumference of each circle to the nearest tenth of an inch. In every case divide the length of the circumference by the length of the diameter and arrange the results in a tabular form as shown below.

Serial Number.	Length of the diameter.	Length of the Circumference.	The number of times the circumference contains the diameter
	A	B	$\left(\frac{B}{A}\right)$
(1)	5 inches.	. ..	....
(2)	8 inches.	.	... .
(3)	10 inches.		.

If you work the last exercise with great care, you will find the numbers in the last column very nearly equal, each being a little greater than 3. This shows that the circumference of *each of these circles* is a little over 3 times the diameter. If you had traced any number of other circles of different radii, and proceeded as in Ex. 3 you would observe the same thing with respect to each of these circles, viz :—the circumference is a little over 3 times the diameter. This would naturally lead you to suspect that perhaps[1] the ratio of the circumference of the diameter is the same for all circles. It has been proved (by a method which you will learn later) that **the ratio of the circumference to the diameter is the same for all circles.**

1 You cannot be *absolutely* sure about it, for, your measurement of the circumference, by sticking pins round it and passing a thread and finally measuring the length of the thread with a scale is after all a crude method, and the results cannot be taken as perfectly accurate.

It has also been found that this ratio is very nearly 22 : 7.[1] Thus we may say that in all circles, the length of the circumference is very approximately $\frac{22}{7}$ times the length of the diameter, or (which amounts to the same thing) $\frac{44}{7}$ times the radius. Thus if you know the radius of a circle you can at once obtain the length of the circumference by multiplying the radius by $\frac{44}{7}$.

Ex. 1. Calculate the lengths of the circumferences of the following circles :—

(i) diameter = 6 inches.

(ii) diameter = 8 inches.

(iii) diameter = 100 yards.

(iv) radius = 1 mile.

Ex. 2. A wheel is of diameter 40 inches. How many revolutions will it make in going over 4 miles ?

Ex. 3. The diameters of a fore-wheel and a hind-wheel of a carriage are 30 inches and 40 inches respectively, find the ratio of their circumferences.

Ex. 4. The diameter of a circular field is half a mile. What will be the cost of fencing it all round at the rate of a rupee for every 5 yards.

Ex. 5. Find to the nearest inch the diameter of a wheel which makes 300 revolutions in rolling over three quarters of a mile.

36. The circular measure of angles—We have seen that in all circles the radius is approximately $\frac{7}{44}$ of the circumference. If the circumference of a particular

1. A more approximate value of this ratio is $\frac{355}{113}$. The Hindu mathematician Aryabhatta (A. D. 476—550) gives the value of this ratio as 62832 : 20000 i.e. 3·1416. The exact value of this ratio cannot be found as a terminating or recurring decimal. Its value correct to 20 decimal places is 3·14159265358979323846.

circle is divided into 44 equal parts and 7 such parts are taken (Fig. 97), we shall get an arc **AB** very nearly equal in length to the radius of this circle.

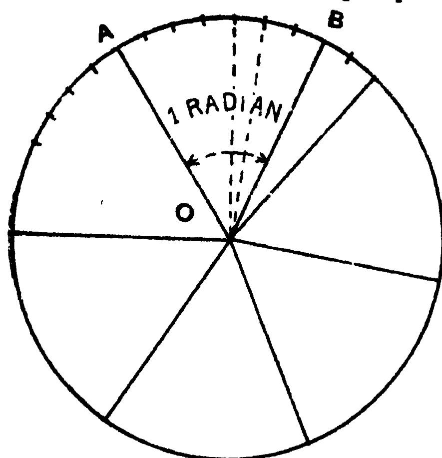


Fig. 97

Each of the 44 equal parts of the circumference subtends an angle of $\frac{360}{44}$ degrees at the centre (§ 31). The arc **AB** consisting of 7 such parts would subtend at the centre an angle 7 times as large. Hence the angle **AOB** $= (\frac{360°}{44} \times 7)$ degrees.

$= 57°$ nearly.

Proceeding in exactly the same way with another circle, you would find that in this circle also, an arc equal to the radius in length, would subtend the same angle (an angle of 57° nearly) at the centre. *This angle which an arc of a circle equal in length to its radius, subtends at the centre of the circle is called a* **Radian.** *The magnitude of this angle is the same for all circles, and is approximately* $\frac{7}{44}$ *of* 360°, or 57° *nearly.*

The radian may be used as a unit of measurement for angles (just like the right angle or the degree), and the measure of the angle in radians is called its **circular measure** (or the radian measure).

The following relations should be noted.

$$\left\{\begin{array}{lcl} 1 \text{ radian} & = & \frac{7}{44} \text{ of 4 right angles.} \\ & = & \frac{7}{11} \text{ of a right angle.} \\ 1 \text{ right angle} & = & \frac{11}{7} \text{ radians.} \end{array}\right.$$

$$\left\{\begin{array}{lcl} 1 \text{ degree} & = & \frac{11}{7 \times 90} \text{ radian.} \\ 1 \text{ radian} & = & \frac{7 \times 90}{11} \text{ degrees.} \end{array}\right.$$

EXERCISE IX.

1. Find the circular measure of the following angles, (i) 1° (ii) 25° (iii) 45° (iv) 120° (v) 2 5 rt. ∠s (vi) $1'$.

2. Express in degrees —(i) 1 radian, (ii) 5 radians, (iii) $\frac{3}{4}$ radian, (iv) 3·2 radians.

3. Find the radian-measure of the angle between the hands of a clock at 4-30 P. M.

4. Arrange the following angles in ascending order of magnitude ;

3·5 radians, 78 degrees and 0·875 right angle.

5. Express in radians the angle subtended at the centre of a circle by an arc, (i) twice as long as the radius, (ii) three times as long, (iii) a third as long, (iv) two-thirds as long, (v) a fifth as long, (vi) two-fifths as long, (vii) three-fifths as long, (viii) four-fifths as long, (ix) seven-eighths as long.

6. Express in radians the angle subtended at the centre of a circle (of diameter 4 inches) by an arc of length, (i) 6 inches, (ii) 5 inches, (iii) 3·5 inches, (iv) 0·6 inch. Express the angles in degrees also.

7. What must be the length of an arc of a circle of diameter 4·5 inches, which subtends at the centre an angle of (i) $\frac{3}{4}$ radian, (ii) $1\frac{2}{3}$ radians, (iii) 50 degrees ?

8. If the radius of a circle is 20 feet, find the length of the arc subtending an angle of 5° at the centre.

CHAPTER II.

RECTILINEAR FIGURES.

37. You have already seen how a circle is a figure bounded entirely by a curved line. You may have figures bounded by straight lines. Such figures are called **rectilinear** (or **rectilineal) figures.** [from Lat. —*rectus* straight, *linea,* a line]. The straight lines which bound a rectilinear figure are called its **sides.** Thus **AB, BC, CD, DA** are the sides of the rectilinear figure **ABCD** (Fig. 98.)

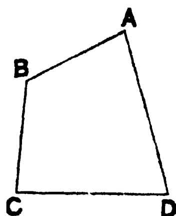
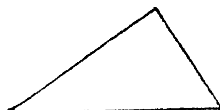
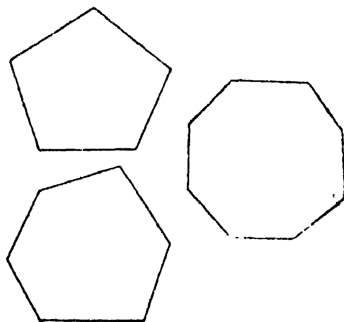


Fig. 98

The simplest rectilinear figure is the **triangle,** bounded by *three* straight lines (Fig. 99). Such a figure, as you see, has *three* angles, hence the name, 'tri-angle' [Lat. *triangulum—tres,* three, *angulus,* an angle]. Can *two straight lines* completely enclose a portion of surface ?


Fig. 99

(A four-sided rectilinear figure is called a **quadrilateral** (Fig. 98) [from Lat.—*quatuor* four, *latus,* a side].) We may have rectilinear figures with 5 sides, 6 sides, etc. Looking at the figures on the margin, you see that each has *as many angles as sides ;* the points where the *consecutive* sides meet (two by two) and where the angles are formed, are called the vertices of the figure.


Fig. 100

Thus in Fig. 98, the vertices are the points **A,B,C,D,** and the angles are **ABC, BCD, CDA, DAB.**

All rectilinear figures with more than four sides (or four angles) are known by the general name, **Polygon** (Lat.—*Polys* many, *gonia* a corner).

Thus we may speak of a 5-sided polygon, 6-sided polygon and so on ; a polygon of *n* sides may be referred to as an *n*-sided polygon. There are specific names for some of these—**pentagon** (a 5-angled i. e. 5-sided figure), **hexagon** (6 sides), **heptagon** (7 sides), **octagon** (8 sides), **nonagon** (9-sides), **decagon** (10 sides), **undecagon** (11 sides), **dodecagon** (12 sides), **quindecagon** (15 sides); we might speak of an '*n*—gon', meaning a polygon of *n* sides.

Note the quadrilateral **PQRS** shown in the margin (Fig. 101). It differs from the quadrilateral **ABCD** in Fig. 98, in this respect that while each angle of the latter is less than a straight angle, the former has an angle (∠**QRS**) which is greater than a straight angle (a *reflex* angle, see note, § 25).[1]

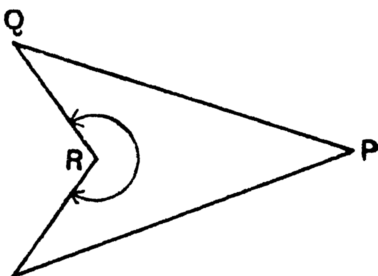


Fig. 101

1. It is interesting to note that Proclus, (410—485 A. D) one of the earlier commentators on Euclid spoke of such a quadrilateral as a four-sided *triangle* (three angled figure), ignoring altogether the angle **QRS**, which happens to exceed two right angles.

If the angles of a polygon are each less than a straight angle, the polygon is said to be convex. The polygons in Fig. 100 are all convex.

EXERCISE X.

N. B. The lengths are to be measured to the nearest tenth of an inch or to the nearest millimetre, and angles to the nearest degree.

1. Measure the sides and angles of the two triangles (Fig. 102). Arrange the results separately for the two triangles.

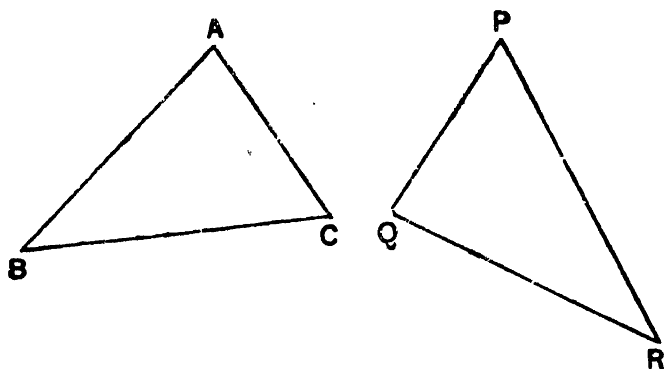


Fig. 102

2. Trace three triangles, and measure their sides and angles. How many sides and angles will you have to measure in all ?

3. Draw a quadrilateral with one of its sides measuring 3 inches. Measure the angles and the remaining three sides of the quadrilateral you have drawn.

4. Draw as neatly as you can a polygon of 6 sides. Take

any point within the figure, and join[1] it to the vertices. How many triangles do you get? Measure all the sides and all the angles of these triangles.

5. Measure the sides and angles of the quadrilaterals shown in Figs. 98, 101, 103

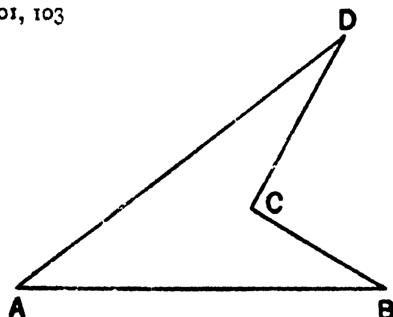


Fig. 103

6. Measure the sides and angles of the polygon shown in Fig. 104

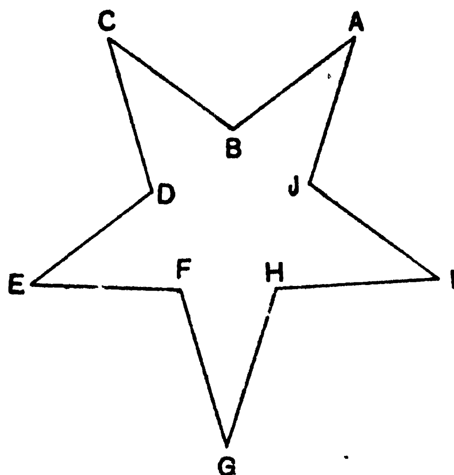


Fig. 104

1. When we speak of two points being joined, we mean that they re joined by a straight line.

38. Points on a circle: points inside and points outside the circle. Take any point O and with it as centre draw a circle of radius 1 inch (Fig. 105). This circle will give you all the points[1] which are at this distance (1 inch) from O. For if you consider any point *not lying on the circle*, you will find that its distance from O cannot be 1 inch; it will be less than 1 inch if the point is inside the circle, and greater than 1 inch if it is outside the circle.

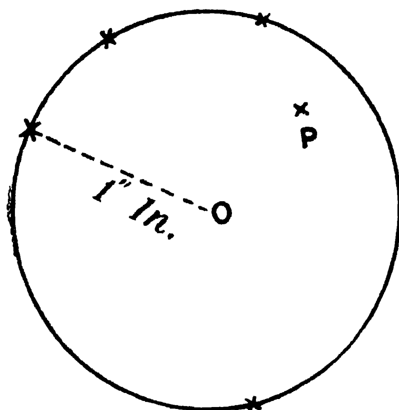


Fig. 105

Mark any point P inside the circle.

(i) Is it true that every point outside the circle is more than 1 inch from P, and that every point more than 1 inch from P, lies outside the circle ?

(ii) Is it true that every point more than 2 inches from P lies outside the circle and that every point outside the circle is more than 2 inches from P ?

1. Obviously on the plane of the paper on which the circle is supposed to be drawn, there are many other points not lying on the plane of the paper which are also at a distance of 1 inch from O. Indeed *all points in space* which are at a distance of 1 inch from O lie on a sphere of centre O and radius 1 inch, the circle on the paper being a section of this sphere.

(iii) Which point on the circle is nearest to the point P, and which point is furthest from P ?

Calculate the distances of these points from P if OP = 0·7 inches.

Ex. 1. A cow is tied to a tree by a rope 100 feet long. Draw a figure to show what ground it can reach (scale, 50 feet to 1 inch).

Ex. 2 A cow is tethered to a post, 10 cubits from the nearest edge (which is straight) of an adjoining rice-field, with a rope 18 cubits long. Draw a figure to show what portion of the field is within reach of the cow ? [Scale 6 cubits to 1 inch.]

39. To find a point which shall be at given distances from two given points. Take two points A and

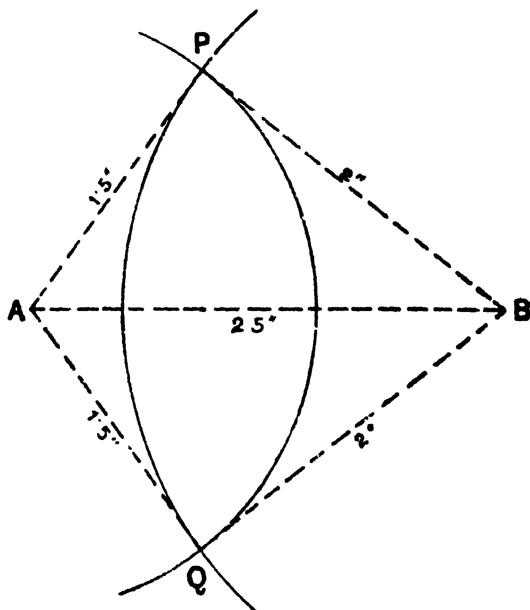


Fig. 106

B 2·5 inches apart, Fig. 106. How to find a point which shall be 1·5 in. from A, and 2 in. from B ? With A as centre draw a circle of radius 1·5 in. and with B as centre draw another circle of radius 2 in. The first circle gives you all the points 1·5 in. from A and the second circle, all the points 2 in. from B. The points P and Q where the two circles intersect, being on both the circles. each of them will be 1·5 in. from A and 2 in. from B.

Ex. 1. Take two points **X** and **Y**, 6 cm. apart, and find a point which is 4 cm. from **X** and 3 cm. from **Y**. How many such points do you get ?

Ex. 2. Take two points **P** and **Q**, 3 inches apart, and try to find a point 1 inch from **P** and 1·5 inch from **Q**. Do you get any such point ?

Ex. 3. Find three points **A**, **B**, **C**, such that **AB**=1·3 in., **BC**=2 in., and **CA**=1·7 in.

Ex. 4. Construct a triangle **PQR**, such that **PQ**=4 cm. **QR**=5 cm. and **RP**=3 cm. Measure the angles.

Ex. 5. Take two points **A** and **B**, and find a point equidistant (i. e equally distant) from them. Find another point equidistant from **A** and **B**. How many such points can you find ? Find 8 different points equidistant from **A** and **B**. Do you find all these points lie on a straight line ?

Ex. 6. It was known that some treasure was hidden in a garden, but the exact spot was not known. The only clue given was that it was at a distance of 8 feet from a certain mango-tree **A**, and 12 feet from a certain guava-tree **B**. How would you discover the treasure from these directions ?

40. If one side of a triangle is equal in length to another side, the triangle is said to be **isosceles** (Gr.—*isos*, equal, *skelos*, a leg).

If all the sides of a triangle are equal in length, the triangle is said to be **equilateral.** (Lat.—*æquus* equal, *latus*, a side).

A triangle in which no two sides are equal is called a scalene triangle [Gr.—*skalenos* —uneven].

In the same way, a polygon is said to be **equilateral** if all its sides are equal; **equiangular**, if all its angles are equal; and **regular,** if all the sides are equal and also all the angles are equal.[1]

41. **Construction of Triangles**:—Go carefully through the following exercise.

EXERCISE XI.

When Three sides are given.

1. Construct triangles **ABC** to the following measurements:

(i)	**BC**=3 in,	**CA**=1·5 in.,	**AB**=2·5 in.;
(ii)	**BC**=2·7 in.,	**CA**=3·2 in.,	**AB**=4 in.:
(iii)	**BC**=5 cm.,	**CA**=6·2 cm.,	**AB**=7 cm;
(iv)	**BC**=2 in.,	**CA**=3 in.,	**AB**=2 in.;
(v)	**AB**=4 cm.,	**BC**=4 cm.,	**AC**=4 cm.;
(vi)	**AB**=5 cm.,	**BC**=4 cm.,	**AC**=4 cm

State which of the triangles are isosceles and which equilateral.

2. Mark three points **P, Q, R.** Joining **PQ, QR, RP,** you get the triangle **PQR.** Construct another triangle **ABC,** such that

1. The student will see (by actual measurement) that an equilateral triangle (i.e, a 3-sided figure) is also equiangular. But if the number of sides exceeds three, a figure *may be equilateral without being equiangular.* The student may be asked to draw some quadrilaterals which are equilateral but not equiangular.

6

AB=**PQ**, **BC**=**QR**, **CA**=**RP**. Measure the angles of the two triangles. Do you find $\angle A = \angle P$, $\angle B = \angle Q$, and $\angle C = \angle R$? Try to express the result in your own words.

3. Construct (or try to construct) triangles to the following measurements :

(i) **AB**=5 cm., **BC**=2 cm., **CA**=6 cm. ;
(ii) **AB**=5 cm., **BC**=2 cm., **CA**=5 cm. ;
(iii) **AB**=5 cm., **BC**=2 cm., **CA**=4 cm. ;
(iv) **AB**=5 cm., **BC**=2 cm., **CA**=3 cm. ;
(v) **AB**=5 cm., **BC**=2 cm., **CA**=2·5 cm. ;
(vi) **AB**=5 cm., **BC**=2 cm., **CA**=2 cm. ,
(vii) **AB**=5 cm., **BC**=3 cm., **CA**=1 cm. ;
(viii) **AB**=5 cm., **BC**=4 cm., **CA**=1 cm. ;

Point out the cases where you do not obtain a triangle. Can you say from the above that for a triangle to be possible it is necessary that **BC**+**CA** should be greater than **AB** ? Can you say that in any triangle, **the sum of any two sides must be greater than the third**. Could you arrive at the last result by considering that *the straight line joining two points is the shortest distance between them* ?

4. Make two triangles each having its sides equal to 2 in., 2·5 in., and 3 in. Measure the angles of the two triangles, and compare the angles of the one with those of the other. Consider Ex. 2 in this connection.

5. Draw two lines **AX**, **AY**, sufficiently long. With centre **A** and any radius draw a circle, and let this circle cut **AX**, **AY**, in the points **B** and **C**, respectively. Join **BC**. Evidently **AB**= **AC**, and the triangle **ABC** is isosceles. Measure the angles **ABC** and **ACB**. Do you find these angles equal ? Formulate the result in your own words.

When two sides and the included angle are given.

6. Make an angle **XAY** equal to 47° (Fig. 107). From **AX** **AY** cut off **AB, AC** equal 4 cm., and 5·2 cm. respectively.

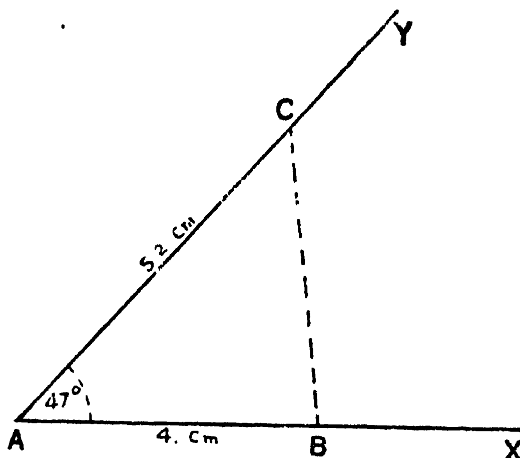


Fig. 107

Join **BC**. You have thus obtained a triangle **ABC**, two of whose sides are 4 cm. and 5 2 cm., and the included angle is 47°. Measure the angles **ABC, ACB**, and also the side **BC**.

7. Construct triangles **ABC** to the following measurements :

(i)	**AB** = 3 in.,	**AC** = 2·5 in.,	∠**A** = 40°.
(ii)	**AB** = 4 cm.,	**AC** = 5 cm.,	∠**A** = 110°.
(iii)	**BC** = 5 cm.,	**CA** = 3·2 cm.,	∠**C** = 85°.
(iv)	**AB** = 1·6 in.,	**BC** = 2·2 in.,	∠**B** = 90°.

8. Mark three points **P, Q, R** Joining **PQ, QR, RP** you get the triangle **PQR**. Construct another triangle **ABC**, such that **AB**=**PQ**, **AC**=**PR**, and ∠**A**=∠**P**. Measure **BC**, ∠**B**, ∠**C**, also **QR**, ∠**Q**, ∠**R**. Do you find **BC**=**QR**, ∠**B**=∠**Q** and ∠**C**=∠**R** ?

Express the results in your own words.

When one side and the angles at its extremities are given.

9. Draw a line **XY** = 3·5 cm. (Fig. 108). From **X** draw a line **XZ** making ∠**YXZ** = 50°; from **Y** draw a line **YZ** making ∠**XYZ** = 72°. The two lines **XZ** and **YZ** meet in **Z**. You have thus obtained a triangle one of whose sides is 3·5 cm., and the angles at the extremities of this sides are 50° and 72°.

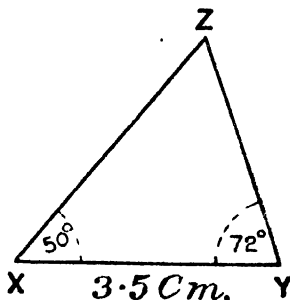


Fig. 108

10. Construct triangles **ABC** to the following measurements

(i) **AB** = 3 in., ∠**A** = 40°, ∠**B** = 75°;
(ii) **AB** = 2·5 in., ∠**A** = 65°, ∠**B** = 90°;
(iii) **BC** = 5 cm., ∠**B** = 30°, ∠**C** = 110°;
(iv) **BC** = 6 cm., ∠**C** = 95°, ∠**B** = 50°;

In each case measure the third angle, and add together the three angles. Do you find the sum = 180° in each case ?

11. Construct or (try to construct) triangles **ABC** to the following measurements :

(i) **BC** = 5 cm., ∠**B** = 70°, ∠**C** = 80°;
(ii) **BC** = 5 cm., ∠**B** = 70°, ∠**C** = 90°;
(iii) **BC** = 5 cm., ∠**B** = 70°, ∠**C** = 100°;
(iv) **BC** = 5 cm., ∠**B** = 70°, ∠**C** = 110°;
(v) **BC** = 5 cm., ∠**B** = 70°, ∠**C** = 120°;
(vi) **BC** = 5 cm., ∠**B** = 70°, ∠**C** = 130°;
(vii) **BC** = 5 cm., ∠**B** = 100°, ∠**C** = 100°:
(viii) **BC** = 5 cm., ∠**B** = 90°, ∠**C** = 90°;
(ix) **BC** = 5 cm., ∠**B** = 90°, ∠**C** = 120°,;

Point out the cases in which you do obtain a triangle, and those in which you do not obtain one. Find $\angle B + \angle C$ in each case. *Do you see that in those cases where* $\angle B + \angle C$ *is less than* 180°, *the triangle is obtained, and that in those cases where* $\angle B + \angle C$ *is greater than* 180°, *the triangle is not obtained? Interpret the cases where* $\angle B + \angle C = 180°$. [see (vi) and (viii).]

When two sides and the angle opposite to one of them is given.

12. To construct a triangle **ABC**, such that **AB** = 2 in., $\angle$ **A** = 30°, and **BC** = 1·2 in.

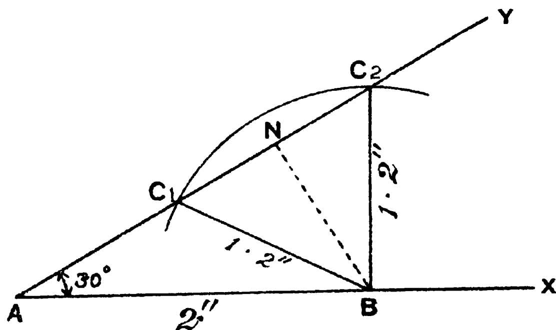


Fig. 109

Make an angle **XAY** (Fig. 109) equal to 30°. From **AX** cut off **AB** = 2 inches. The vertex **C** of the required triangle must be a point on the line **AY**; also because the side **BC** is to be 1·2 in., the point **C** must be 1·2 in. from **B**, hence it must lie on the circle constructed with **B** as centre and 1·2 in. as radius. This circle, as you see in the figure, cuts **AY** in *two* points C_1, C_2. Thus *two* triangles ABC_1, ABC_2 are obtained, each having the given parts.

Try the following cases :—

(i) AB = 2 in., ∠A = 30°, BC = 1·5 in. ;
(ii) AB = 2 in., ∠A = 30°, BC = 1·7 in. ;
(iii) AB = 2 in., ∠A = 30°, BC = 2 in. ;
(iv) AB = 2 in., ∠A = 30°, BC = 2·3 in. ;
(v) AB = 2 in., ∠A = 30°, BC = 2·5 in. ;
(vi) AB = 2 in., ∠A = 30°, BC = 1·3 in. ;
(vii) AB = 2 in., ∠A = 30°, BC = 1·1 in. ;
(viii) AB = 2 in., ∠A = 30°, BC = 1 in. ;
(ix) AB = 2 in., ∠A = 30°, BC = 0·9 in. ;
(x) AB = 2 in., ∠A = 30°, BC = 0·7 in. ;
(xi) AB = 2 in., ∠A = 30°, BC = 0·5 in.

From the above, point out the cases, (i) in which you get *two* distinct triangles with the given parts, (ii) in which you get only *one* triangle, (iii) in which *no* triangle is obtained.

Draw the perpendicular **BN** upon **AY** (Fig. 109) and measure its length. Do you find it to be 1 inch ?

Point out the cases where **BC** *is greater than* **BN**. *Do you get a triangle in each of these cases, if so, how many ? Point out the case where* **BC = BN**, *do you get a triangle in this case, if so, how many ?*

Point out the cases where **BC** *is less than* **BN**. *Do you get a triangle in any of these cases ?*

Point out the cases in which **BC > AB**. *How many triangles do you get in each of these cases ?*

Point out the case in which **BC = AB**. *How many triangles do you get in this case ? Point out the cases in which* **BC** *is less than* **AB** *but greater than* **BN**. *How many triangles do you get in each of these cases ?*

Note. The exercise given above is extremely important, and the student must patiently go through the whole of it He will then be

in a position to construct triangles when **three of the six parts** are assigned in any of the following ways, (i) *three sides*, (ii) *two sides and the included angle*, (iii) *one side and the angles at its extremities, and* (iv) *two sides and the angle opposite to one of them.*

42. Construction of Quadrilaterals.

EXERCISE XII.

1. Construct a quadrilateral **ABCD** such that **AB** = 2", ∠**B** = 40°, **BC** = 1·5", ∠**C** = 120°, **CD** = 2·3". Measure the side **AD**, and the ∠s **A** and **D**. Measure also the *diagonals* **AC**, **BD**.

Note. The straight line joining any corner of a quadrilateral to the opposite corner is called a **diagonal**. Thus every quadrilateral has two diagonals.

2. Construct a quadrilateral **ABCD**, such that **AB** = **CD** = 6 cm., ∠**B** = ∠**C** = 120°, **BC** = 4 cm. Measure **AD**, and ∠s **A** and **D**; do you find these angles equal? Measure the diagonals **AC**, **BD**; do you find them equal?

3. Construct a quadrilateral **ABCD**, such that the diagonal **AC** = 5 cm., and the sides **AB**, **BC**, **CD**, **DA** = 3 cm., 4 cm., 2·5 cm., 4·3 cm., respectively. Measure the length of the other diagonal (**BD**).

4. Construct another quadrilateral with sides of the same lengths as in Ex. 3, but the diagonal **AC** measuring 3 cm.

5. Construct a quadrilateral **ABCD** whose diagonals **AC**, **BD** are 3 in. and 2 in. respectively, bisect each other, and include an angle of 40°. Measure the sides and angles of the quadrilateral you construct. Do you find the opposite sides and opposite angles equal?

6. Construct quadrilaterals **ABCD** to the following measurements:

(i) **AB** = **BC** = **CD** = **DA** = 1·5 in., **BD** = 2 in. (measure **AC**);

(ii) AB = BC = CD = DA = 1·5 in., BD = 1 in., (measure AC);

(iii) AB = 2 in., BC = 1·5 in., AC = 1·5 in., ∠A = ∠C = 100°;

(iv) OA = 4 cm., OB = 3·4 cm., OC = 2 cm., OD = 5 cm., AB = 3 cm. (O being the point where the diagonals intersect);

(v) AB = 2 in., BC = 2·5 in., AD = 1·8 in., AC = 3·2 in., BD = 2·5 in.

(vi) AB = 5 cm., AD = 6 cm., ∠A = 50°, BC = 4·5 cm., CD = 7 cm.

(vii) AB = BC = CD = DA = 2 in., ∠A = a right angle (measure the other angles);

(viii) AB = BC = CD = DA = 2 in., ∠A = 120° (measure the other angles);

(ix) AB = CD = 6 cm., BC = AD = 4 cm., ∠B = 70°.

CHAPTER III.

GEOMETRICAL TRUTHS AND SIMPLER THEOREMS AND PROBLEMS.

43. Take two intersecting straight lines **AOB, COD** (Fig. 110) cutting each other at the point **O**. If you

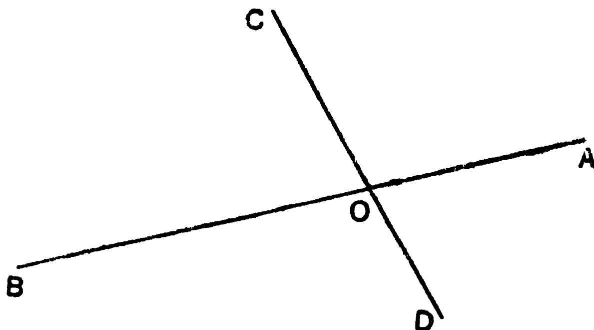


Fig. 110

observe carefully the four angles formed at **O**, you will very likely guess the *vertically opposite angles* (§ 20) **AOC** and **BOD**, to be equal; likewise you will guess the other pair of vertically opposite angles, **AOD, BOC,** also to be equal. You will naturally proceed to test your guess by actually measuring the angles with a protractor or comparing them directly by cutting out one of the angles and applying it to the other.

Actually measure the angles **AOC, BOD** with a protractor. Do you find the measures equal (or nearly equal)?

Measure the other pair of vertically opposite angles, **AOD, BOC.** Do you find the measures equal?

Take some more pairs of intersecting lines as shown in Figs. 111, 112, and compare the vertically opposite angles in each case, by measuring them with a protractor as before.

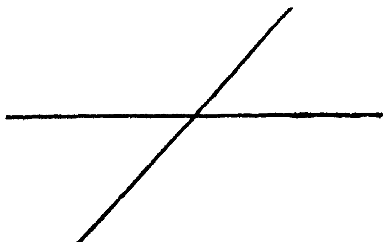


Fig. 111

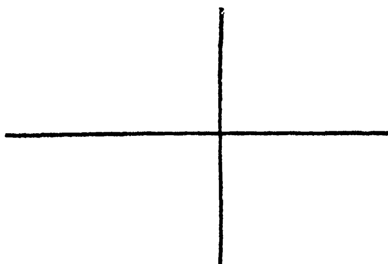


Fig. 112

If you take the measurements with great care, you will find that the vertically opposite angles are equal, in each case. These experiments will thus suggest to you a general truth:—'*if two straight lines intersect, the vertically opposite angles are equal.*'

The way in which you have arrived at this truth is purely *experimental.* What you have done really amounts to your verifying the truth in a number of particular cases, and then *assuming* that it will hold in *all* possible cases. This assumption is in itself a drawback; one may raise the objection 'yes, as accurately as we can measure, we see that *in the cases actually examined,* the vertically opposite angles are found to be equal, but who knows there may not be cases in which the vertically opposite angles are *not* equal?' How are you to answer him, or be convinced yourself, that the vertically opposite angles

are equal in *every* case of two intersecting straight lines? You cannot possibly measure *every* case, or there might be cases where practical difficulties might stand in the way of taking measurements. Moreover all measurements are more or less approximate in character. No measuring instrument can be ideally perfect in its construction, nor can it measure sufficiently small quantities or residues of quantities; the graduations have a certain amount of breadth, and the eye cannot be trusted to make perfectly accurate judgment.

44. Take two long thin strips of wood and pin them together as shown in Fig. 113. Bring the strip

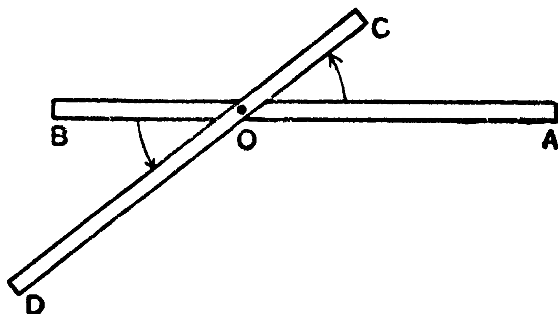


Fig. 113

COD into coincidence with **AOB** in direction. If you now turn the arm **OC** through any angle **AOC**, the other arm **OD** will at the same time turn round **O** tracing out the angle **BOD**. As **OD** rotates through the same amount as **OC**, the angles **AOC** and **BOD** are equal.

This establishes the truth on a universal basis, and convinces you as to why the vertically opposite angles must be equal in *every* case of two intersecting straight lines.

A more scientific way would be to reason in the following way—

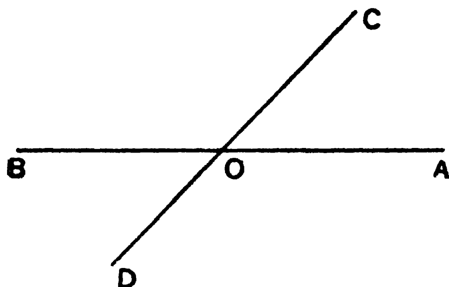


Fig. 114

$\angle$**AOC** + $\angle$**COB** = a straight angle;
also $\angle$**COB** + $\angle$**BOD** = a straight angle;
$\therefore$ $\angle$**AOC** + $\angle$**COB** = $\angle$**COB** + $\angle$**BOD**.

Taking away the common angle **COB** from both sides, we have $\angle$**AOC** = $\angle$**BOD**.

Note that in this proof we have made use of the following truths :—

(i) *If a straight line* (**OC**) *stands upon another straight line* (**AOB**), *the sum of the adjacent angles* (**AOC, COB**)=*a straight angle or two right angles.* This truth was arrived at before (§ 26, Note II.)

(ii) *Things which are equal to the same thing are equal to one another*; the sums $\angle$**AOC** + $\angle$**COB** and $\angle$**COB** + $\angle$**BOD** being each equal to a straight angle, are themselves equal.

(iii) *If equals be taken from equals, the remainders are equal*; Thus $\angle$**AOC** + $\angle$**COB** and $\angle$**COB** + $\angle$**BOD** being equal, if we take away the same angle **COB**

from both, the remainders, the $\angle^s$ **AOC**, **BOD**, will be equal.

The truths (ii) and (iii) are evident (no body thinks of questioning them) and are called *Axioms*.

45. **Another example of a geometrical truth**—You know how to construct a triangle, the length of one side and the angles at the extremities of this side, being given (Ex. 9 p. 84). Let us suppose that one side is to be 2 in. and the angles at the extremities of this side to be 30° and 50°.

Construct a triangle **ABC** such that

BC = 2 in.,
∠**B** = 30°
∠**C** = 50°

Construct another triangle **PQR**,

such that **QR** = 2 in.,
∠**Q** = 30°
∠**R** = 50°,

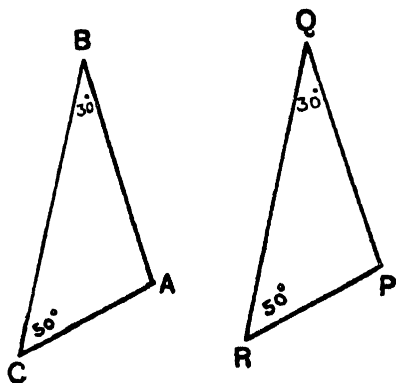


Fig. 115

The two triangles **ABC**, **PQR** have

side **QR** = side **BC** (each being 2 in.),
∠**Q** = ∠**B** (each being 30°),
∠**R** = ∠**C** (each being 50°).

You will find the triangles so similar in shape and size, that you will naturally guess

side **QP** = side **BA**,
side **RP** = side **CA**,
∠**P** = ∠**A**,

and will verify your guess by actual measurements.

A few experiments of this type will lead you to arrive at a general truth—*If two triangles are such that a side, and the angles at the extremities of this side, of one triangle, are respectively equal to a side, and the angles at the extremities of that side, of the other triangle, then the remaining parts of the one will also be equel to the remaining parts of the other, each to each*

The manner in which you have arrived at this truth is purely experimental and as such has defects already pointed out. *A more scientific way* would be to reason in the following way.

If the triangle **ABC** were cut out and placed on top of the triangle **PQR** so that **BC** exactly fitted **QR** (this would be possible because **BC** and **QR** are equal in length), then because $\angle B = \angle Q$, **BA** would lie along **QP** and because $\angle C = \angle R$, **CA** would lie along **RP**. Thus the triangle **ABC** would exactly fit the triangle **PQR**, hence $\angle A = \angle P$ and **BA** = **QP**, and **CA** = **RP**.

46. **A third example**—Take a line **BC**, at **B** draw a line **BA** making the angle **ABC** = 40° (say), and at **C** draw a line **CA** making the angle **ACB** = 80° (say). This construction *fixes* the angle formed at **A**. If you measure this angle carefully you will find it to be 60°. Now add together the three angles **A, B, C** and you find the sum to be 180° or 2 right angles.

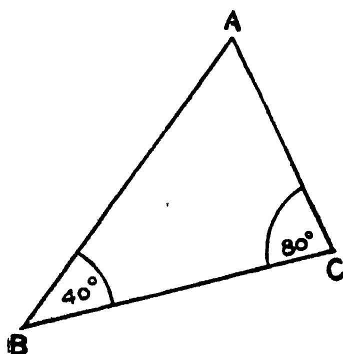


Fig. 116

Draw another triangle, marking any three points **P,Q, R** and joining them two by two (Fig. 117). Measure the

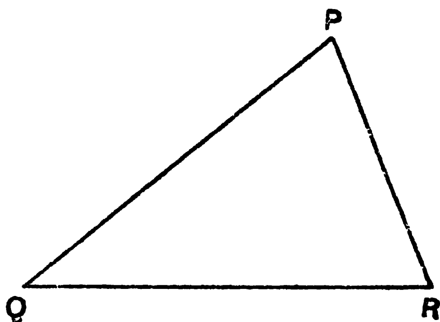


Fig. 117

three angles at **P, Q, R,** and add the measures. Do you find the sum = 180° ?

Cut up the angles and place them together as shown in Fig. 118. You find that they make up a straight angle.

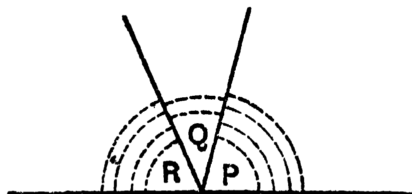


Fig. 118

Examining a few more triangles, you come to the conclusion, '*The three angles of a triangle are together equal to two right angles.*' This is another general truth, and you have arrived at it in an experimental way. A scientific proof will be given later.

47. Scientific proof as distinct from Experimental Verification. The Science of Geometry.

From the illustrations given above you will come to understand the distinction between what has been called *a scientific proof* of a truth, and the merely *experimental verification* of a truth. The superiority of the former over the latter is manifest. But this should not lead you to belittle the importance of the experimental verification of a truth, the clue to which, in the first stages of the growth of a science, is generally supplied by observation, or experience or intuition, combined with reflection.[1] Scientific proof comes later and proceeds from the necessity of systematising the stray, and hitherto unconnected pieces of knowledge or truths into a connected whole, in which the individual truths are presented as so many links of a chain, each truth depending upon one or more of the preceding truths. For this purpose it is necessary to start with certain fundamental truths, called axioms, which are self-evident and taken for granted. Some of these are stated below.

Axioms of magnitude in general *(common notions).*

I. *Things which are equal to the same thing are equal to one another.*

II. *If equals are added to equals the wholes are equal.*

---

1. How the necessity of doing a thing may lead to the discovery of a geometrical truth is illustrated in Ex. 3 p 82, where it is shown how the truth 'two sides of a triangle are together greater than the third side' is discovered from the attempt to construct triangles with three sides assigned. Ex. 11 p. 84, suggests that the angles of a triangle are some how connected with one another.

III. *If equals are taken from equals, the remainders are equal.* [If $\angle$**A**$=\angle$**B**, then $180° - \angle$**A**$=180° - \angle$**B**, this leads to the truth that supplements of equal angles are equal.]

IV. *The same multiples* (*or fractions*) *of equals are equal.* [If $\angle$**A**$=\angle$**B**, then twice $\angle$**A**$=$twice $\angle$**B**, or half of $\angle$**A**$=$half of $\angle$**B**.]

V. *The whole is greater than a part*

Axioms peculiar to geometry.

I. *Things* (lines, angles, or figures) *which coincide are equal to one another.* [for an application of this axiom see § 45.]

II. *The straight line is the shortest distance between the two points which it joins.* [Thus, two sides of a triangle are together greater than the third ; an arc of a circle is greater in length than the corresponding chord.]

III. *If two straight lines have two points in common, then they coincide entirely.*

This list does not comprise all the axioms which will be assumed subsequently in the treatment of the subject.

48. **A few remarks on straight lines** :—Every straight line may be regarded as extending without limit in either direction. It is then called an **indefinite or unlimited straight line.**

Sometimes a straight line is regarded as terminating at a point in one direction, and extending indefinitely in the other direction. The two arms of an angle are supposed as terminating at the vertex.

Sometimes we have to consider a straight line included between two end-points. Such a portion may be called a **finite straight line,** or simply a **segment.** The sides of a triangle are finite straight lines.

Any finite straight line may be supposed to be **produced** either way to any extent. A finite straight line being traced on paper you may *produce* it in either direction with the help of a ruler.

49. *Meanings of certain terms to be used subsequently.*

Anything that is proposed for discussion may be called a **Proposition.**

In Geometry, propositions are of two kinds, *Theorems* and *Problems.*

A **Theorem** is a proposition which requires a certain geometrical truth to be proved.

A **Problem** is a proposition which requires some geometrical construction to be performed (e.g., bisection of an angle, drawing of a perpendicular from a given point to a given straight line, construction of an equilateral triangle on a given straight line).

50. The construction that may be required in a problem, or in the proof of a theorem is restricted to the drawing of straight lines and circles, and this is to be done *with ruler and compasses* only.[1]

---

1. "The ideal of Greek Geometry may fairly be described as *construction under self-imposed definite limitations*', ... All constructions were to be reduced to the use of ruler and compasses only—the respective concrete embodiments of the ideal straight line and circle." [Branford—*Mathematical Education*, p 338.] So great were the restrictions placed upon the use of instruments (borrowed from

51. The following abbreviations and symbols will be used,

$\therefore$ (therefore)
$\because$ (because)
$=$ (equal to)
$>$ (greater than)
$<$ (less than)

$\angle$ (angle)
rt. $\angle$ (right angle)
st. line (straight line)
Vert. opp. (vertically opposite)

perp. or $\perp$ (perpendicular)
$\parallel$ (parallel)
$\triangle$ (triangle)
$\odot$ (circle)
$\odot^{ce}$ (circumference)

A few more abbreviations will be introduced later on.

In numerical examples the following abbreviations will be used,

12° 5′ 20″ (12 degrees, 5 minutes, 20 seconds).

5 ft. 6 in. or 5′ 6″ (5 feet 6 inches.)

2 m. 3 cm. 4 mm. (2 metres, 3 centimetres, 4 millimetres.

---

mechanical art) that even in the use of compasses, it was enjoined that the distance *must not be carried*, lest the compasses close of themselves the moment they loose contact with the paper.

THEOREM 1. [Euclid 1, 13].[1]

52. **If a straight line stands upon another, the sum of the adjacent angles (see § 20) so formed is two right angles.**

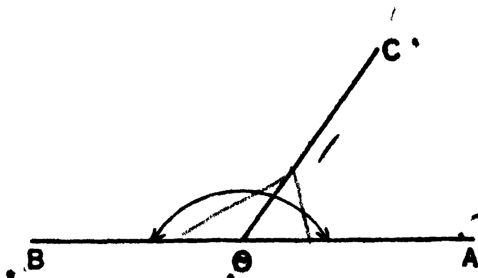


Fig. 119

The st. line OC standing upon another st. line, AB, forms with it adjacent angles AOC, BOC.

To prove that ∠AOC + ∠BOC = 2 right angles.

Proof :— ∠AOC + ∠BOC = ∠AOB
which is a straight angle, for AOB is a straight line. [§ 22.]

But a straight angle = 2 rt. ∠s.

Hence ∠AOC + ∠BOC = 2 rt. ∠s.

Note 1. The adjacent ∠s AOC, BOC are supplementary (§ 26).

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Note 2. If $\angle AOC = \angle BOC$, each of them is a right angle (Fig. 120) because their sum is two right angles.

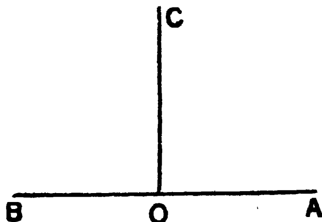


Fig. 120

Thus if a straight line OC standing upon another line AB, makes the adjacent angles (AOC, BOC) equal, then these angles are right angles, and OC is perpendicular to AB.

Note 3. The following truths should be noticed.

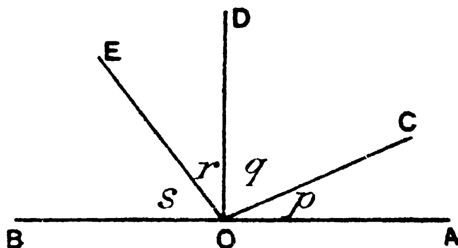


Fig. 121

(a) The sum of all the angles formed at a point **O** on one side of a straight line **AB** = 2 right angles.

Thus $\angle p + \angle q + \angle r + \angle s = 2$ right angles (Fig 121).

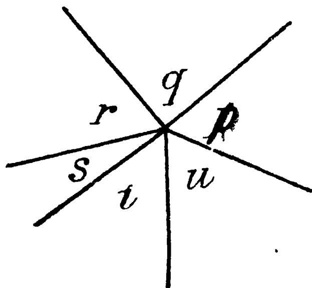


Fig. 122

(b) If a number of straight lines are drawn from a point, the sum of all the angles around the point = 4 right angles. Thus $\angle p + \angle q + \angle r + \angle s + \angle t + \angle u = 4$ right angles (Fig. 122).

EXERCISE XIII.

1. What is the quickest way of drawing the supplement of an angle traced on paper ?

2. Make an angle **ABC** = 40°, and produce **AB** to **D** and **CB** to **E**. Calculate the angles **CBD** and **ABE**. Check by measurement.

3. If in the last example ∠**ABC** were 85°, what would the angles **CBD** and **ABE** be ? Draw a figure and measure to check your calculation.

4. If any point **O** within a convex pentagon be joined to the vertices **A**, **B**, **C**, **D**, **E**, show that the sum of the ∠s **AOB**, **BOC**, **COD**, **DOE**, **EOA** is four right angles

5 The sum of the angles which the sides of a convex polygon of any number of sides subtend at any point within the polygon is four right angles.

6. **POQ**, **QOR**, **ROP** are three angles at **O**, and ∠**POQ** = 140°, ∠**QOR** = 100°, what is the size of the ∠**ROP** ? Test your result by drawing a figure and measuring the angle

7 Revise Exs 7—23 (Exercise VI) and Exs. 27, 28 (Exercise VIII)

8. **ABC** *is a triangle such that* ∠**ABC** = ∠**ACB**, *if* **BC** *is produced both ways, show that the exterior angles so formed are equal.*

9 **ABC** is a triangle such that ∠**ABC** = ∠**ACB** ; if **AB** is produced to **D** and **AC** to **E**, show that the ∠s **CBD** and **BCE** are equal

10. If two lines cross each other, and one of the four angles formed at the point of intersection be a right angle, then the remaining three angles will also be right angles.

11. **If a straight line OC stands upon another straight line AOB, then the bisectors (OP, OQ) of the adjacent angles (AOC, COB) are at right angles (Fig. 123).**

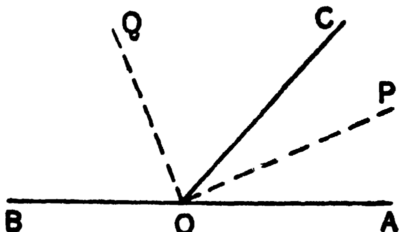


Fig. 123

Note. The lines which bisect an angle and the adjacent angle obtained by producing one of its arms, are called the internal and external bisectors of the angle respectively.

In Fig 123, **OP** and **OQ** are respectively the internal and external bisectors of the angle **AOC**; and **OQ**, **OP** are respectively the internal and external bisectors of the ∠**BOC**. We see that **the internal and external bisectors of an angle are at right angles.**[1]

12. In Fig. 123, show that (i) ∠s **AOP, BOQ** are complementary (ii) ∠s **COQ, AOP** are complementary (iii) ∠s **BOP, COP** are supplementary (iv) the ∠s **AOQ, COQ** are supplementary.

---

1. If **CO** were produced to **D**, we should have the lines **AOB, COD** crossing each other at **O**. If **OP, OQ, OP′, OQ′** are the bisectors of the ∠s **AOC, COB, BOD, DOA**, then ∠**POQ** is a right angle (Ex. 11), similarly ∠s **QOP′, P′OQ′, Q′OP** are right angles. The ∠s **QOQ′, POP′** being each two right angles, are straight angles. Hence **QOQ′, POP′** are two *straight lines*, (See Theor. 2, § 53). **QOQ′** is the internal bisector of each of the ∠s **COB, DOA** and external bisector of each of the ∠s **AOC, BOD**. Similarly **POP′** is the internal bisector of each of the ∠s **AOC, BOD**, and the external bisector of each of the ∠s **BOC, AOD**.

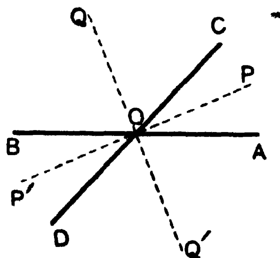


Fig. 123 a.

THEOREM 2. [Euclid 1, 14.]

53. If the sum of two adjacent angles is equal to two right angles, the exterior arms of the angles lie in one straight line.

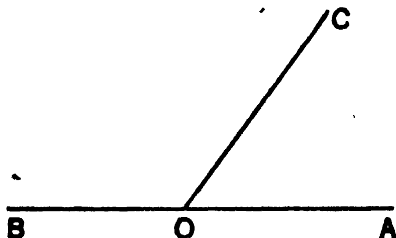


Fig. 124

It is given that

$\angle$**AOC** + $\angle$**BOC**=2 rt. $\angle^s$,

To prove that **OA, OB** are in one straight line.

Proof :—$\angle$**AOC** + $\angle$**BOC** = 2 rt. $\angle^s$. *(Given)*

But $\angle^s$ **AOC, BOC** together make the $\angle$**AOB**.

$\therefore$ $\angle$**AOB** = 2 rt. $\angle^s$ = a straight angle.

$\therefore$ **OA and OB are in one straight line.** [See § 22.]

Ex. 1. **If from a point O of a straight line OC, two straight lines OA, OB are drawn at right angles to OC on opposite sides of it, then AOB will be one straight line**

Ex. 2. Make three pairs of adjacent angles as indicated below : (i) 70°, 110°, (ii) 45°, 135°, (iii) 85°, 95°, and verify in each case that the exterior arms are in one straight line.

Note.—In Theorem 1, it is given that **OA**, **OB** are in one straight line, and it is proved that the sum of the adjacent angles **AOC BOC**=2 rt. $\angle^s$.

In Theorem 2, it is given that the sum of the adjacent angles **AOC, BOC** = 2 rt. ∠s, and it is proved that **OA, OB** are in one straight line.

In every Geometrical Theorem, we suppose certain conditions to be satisfied; and we conclude or prove certain results as following from those conditions. The conditions supposed to be satisfied or given, constitute what is called the **Hypothesis**, and the result finally arrived at, constitute what is called the **Conclusion**.

It is readily seen that the hypothesis and conclusion of Theorem 2, are respectively the conclusion and hypothesis of Theorem 1. *If two theorems are such that the hypothesis and conclusion of the one are respectively the conclusion and hypothesis of the other, one is said to be the* **converse** *of the other.*

Theorems 1 and 2 are evidently converse theorems.[1]

EXERCISE XIV.

1. If from the point **O** of a straight line **OA**, straight lines **OB, OC** are drawn on opposite sides of it, making the angles **AOB, AOC** equal to 47° and 118°, will the lines **OB, OC**, be in one straight line?

2. If the bisectors of two adjacent angles be at right angles then the exterior arms of the two angles will be in one straight line.

3. **POQ, QOR, ROS** are three angles at **O**. If ∠**POQ** = 40°, ∠**QOR** = 80°, and ∠**ROS** = 60°, will **OS, OP** lie in one straight line?

---

1. If a statement is true, the converse statement is not always true. See § 38. query, (ii).

THEOREM 3. [Euclid 1, 15].

54. **If two straight lines intersect the vertically opposite angles (§ 20) are equal.**

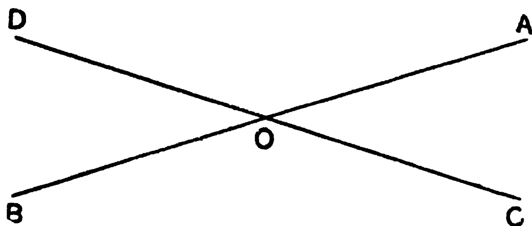


Fig. 125

The st. lines **AB** and **CD** intersect at **O**.

To prove that ∠**AOC** = ∠**BOD** and ∠**AOD** = ∠**BOC**.

Proof :—Because **CO** meets **AB** at **O**.

∴ ∠**AOC** + ∠**COB** = 2 rt. ∠ˢ. (*Theor.* 1.)

Because **BO** meets **CD** at **O**.

∴ ∠**BOD** + ∠**COB** = 2 rt. ∠ˢ.

Hence ∠**AOC** + ∠**COB** = ∠**BOD** + ∠**COB**.

From these equals take ∠**COB**,

∴ ∠**AOC** = ∠**BOD**.

Similarly it may be shown that ∠**AOD** = ∠**BOC**.

Ex. Verify the truth of the Theorem by actually measuring ∠ˢ **AOB, BOD, AOD, BOC**. Draw three more pairs of intersecting lines, and verify the truth in each case by measurement.

Note. From a point O in the straight line AB, two lines OC, OD are drawn on opposite sides of it, such that ∟AOC=∟BOD. Show that OC and OD are in one straight line. [∟BOD+∠BOC=∠AOC+∟BOC=2 right angles. (Theor. 1). ∴ OC, OD are in one straight line. (Theor. 2.)]

EXERCISE XV.

1. Make an angle **AOC** equal to 50°, and produce **AO** to **B** and **CO** to **D**. Write down the values of the ∠s **BOC, BOD**, and **AOD**; and verify by measurement.

2. If ∟**AOC**+∟**BOD**=150°, where **AB, CD** are two straight lines intersecting at **O**, calculate the ∠s **AOC, COB, BOD, DOA**.

3. **AB, CD** are two straight lines intersecting at **O**. If ∠**AOC**+∠**COB**+∟**BOD**=220°, find by calculation the angles **AOC, COB, BOD, DOA**.

4. Three straight lines **AB, CD, EF** pass through the same point **O**. Arrange the six angles formed at **O** in pairs of equal angles.

5. **AB, CD** intersect at **O** and **OX** is the bisector of the ∟**AOC**, show that **XO** produced, bisects the angle **BOD**.

6. *The bisectors of a pair of vertically opposite angles are in one straight line*

55. If two figures can be made to coincide (*i. e.* fit exactly) by superposition, they are said to be **congruent** or **identically equal** [from Lat,—*congruere*, to run together]. In § 45 it has been shown how the triangle ABC might be placed upon the triangle PQR so as to fit it exactly; the triangles ABC and PQR are therefore congruent. It should be observed that if two triangles are congruent the three sides and the three angles of the one are equal respectively to the three sides and the three angles of the other; also the triangles enclose equal amounts of surface, that is to say, they are equal in area.

THEOREM 4. [Euclid 1, 4].

56. **If two triangles have two sides of the one, equal to two sides of the other each to each, and also the angles contained by those sides equal, the triangles are congruent.**

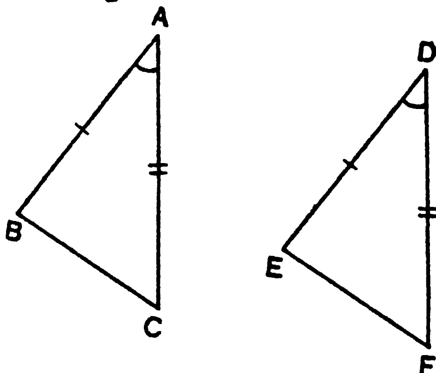


Fig. 126

ABC and **DEF** are two triangles,
such that **AB** = **DE**,
AC = **DF**,
and the included ∠**A** = included ∠**D**.

To prove that △[s] **ABC, DEF** are congruent.

Proof :—If the △ **ABC** were placed on top of the △ **DEF** so that *A fell on D*, and **AB** along **DE**,

then because **AB** = **DE**, *the point B would fall on the point E.*

Again because ∠**A** = ∠**D**, **AC** would fall along **DF** ; and because **AC** = **DF**, *the point C would fall on the point F.*

As **B** falls on **E** and **C** on **F**, the side **BC** will coincide with the side **EF** [see III under Axioms peculiar to geomet y, p. 97].

Thus *the △ ABC would exactly fit the △DEF*,

Hence **the** △[s] **ABC, DEF are congruent.**

Note 1. From the fact of **AB** being equal to **DE, AC** being equal to **DF**, and the included **∠A** being equal to the included **∠D**, is deduced the perfect equality of the two △s **ABC, DEF**, so that the remaining three parts of the △ **ABC** are equal respectively to the remaining three parts of the △ **DEF**, *that is*,

side **BC** = side **EF**,
angle **B** = angle **E**,
and angle **C** = angle **F**.

When two triangles are congruent, all the six parts of the one are equal to the six parts of the other, and the two triangles are said to be **equal in all respects**[1]; and to every part of one triangle is said to **correspond** that part of the other to which it is equal. Thus to the sides **AB**, **BC**, **CA** of the △ **ABC** correspond respectively the sides **DE**, **EF**, **FD** of the △ **DEF**, and to angles **A, B, C** correspond respectively the angles **D, E, F**. It may also be observed that the angles opposite to a pair of corresponding sides in the two triangles, are corresponding angles, and the sides opposite to a pair of corresponding angles are corresponding sides.

Note 2. If you are asked to construct a triangle, *two sides and the included angle being assigned*, you will proceed as in Ex. 6. p. 83.

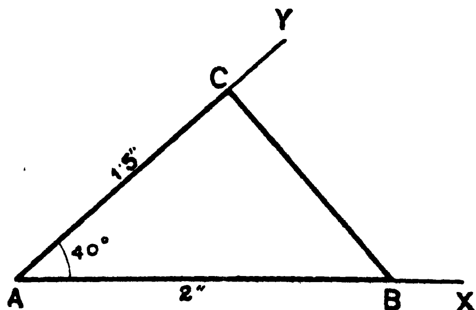


Fig. 127 (a)

Suppose it is required to construct a triangle **ABC**, such that **AB** = 2 in. **AC** = 1.5 in., and the included **∠A** = 40°. You will start by placing on the paper an angle **XAY** = 40° [Fig. 127 (a)] and then cut off from the arms **AX**, **AY**, two lengths **AB**, **AC** equal respectively to 2 inches

1. Two triangles may be equal in area without being equal in all respects.

and 1·5 inches. Joining **BC** you will obtain a triangle **ABC** with the given parts. Once you have placed the ∠**XAY** of size 40°, the rest of the construction is *fixed* by the assignment of the lengths of the sides **AB**, **AC**. If you had started by placing an angle of 40° in a different part of the paper, and done the rest of the construction as before, you would get another triangle **A′B′C′** [Fig. 127 (b)] with the same three assigned parts. From the very manner of construction of the two triangles **ABC**, **A′B′C′**, you will see that this construction is unique, and that though you have constructed two triangles in different parts of the paper, they are exactly *alike in size and shape*, and are really one and the same triangle only placed in different positions, and one could be made to fit the other exactly. In other words, the ∠s **ABC**, **A′B′C′** are *equal in all respects*; and it is this truth which is established on a universal basis in Theorem 4. Here two side and the included angle are assigned, you may construct as many triangles as you like with the given parts, but all these triangles will only be different positions of a **triangle of unique size and shape**.

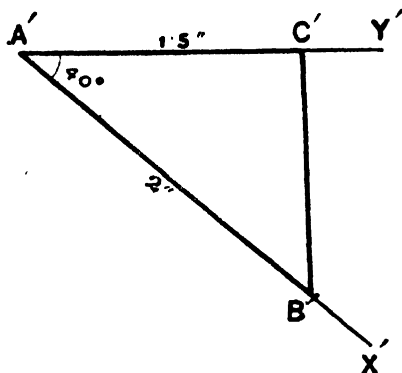


Fig. 127 (b)

Note 3. Observe the two congruent triangles **ABC**, **DEF**, occupying positions as shown in Fig. 128. In order to make one of the △s fit the other, you will have to **turn it over** before you place it on top of the other.

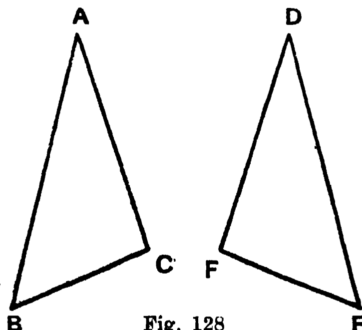


Fig. 128

EXERCISE XVI.

1. Draw two triangles ABC, $A'B'C'$ such that $AB = A'B' = 4$ cm., $AC = A'C' = 5$ cm., $\angle A = \angle A' = 70^\circ$. Verify by measurement that $BC = B'C'$, $\angle B = \angle B'$, $\angle C = \angle C'$. Arrange the 12 parts of the two triangles in 6 pairs of corresponding parts

2. Draw three triangles ABC, $A'B'C'$, $A''B''C''$ such that $BC = B'C' = B''C'' = 2{\cdot}3$ in., $BA = B'A'$ $B''A'' = 3$ in., and $\angle B = \angle B' = \angle B'' = 55^\circ$. Verify by measurement that (i) $AC = A'C' = A''C''$ and (ii) $\angle A = \angle A' = \angle A''$, and $\angle C = \angle C' = \angle C''$.

3. ABC is an isosceles triangle such that $AB = AC$; show that any point P on the bisector of the angle BAC is equidistant from B and C (i e. $BP = CP$).

4. $ABCD$ is a square, O being the mid-point of the side AB; show that $OC = OD$ and $\angle AOD = \angle BOC$.

Note. A **square** is a 4-sided figure with all its sides equal and all its angles right angles [see Fig. 218].

5 In Ex. 4, equal lengths AX, BY are cut off from the sides AD, BC respectively. Show that $OX = OY$ and $\angle DOX = \angle COY$. Show also that $\triangle XOD = \triangle YOC$,

6. *Show that in an isosceles triangle, the bisector of the angle included by the equal sides, bisects and is perpendicular to the third side (i.e., the side opposite to this angle).*

7. ABC is an isosceles triangle in which the equal sides, AB, AC are produced to P and Q, respectively. If $BP = CQ$, show that $CP = BQ$.

8. AB is a straight line, and O is its mid-point; through O a perpendicular is drawn to AB; show that any point on this perpendicular is equidistant from A and B.

9. $ABCD$ is a quadrilateral such that $AD = BC$ and $\angle A = \angle B$; O is the mid-point of AB. Show that $OC = OD$; show also that the $\angle$s AOC, AOD are supplementary.

10. **ABC** is any triangle, and **D** is the mid-point of **BC**. Join **AD** and produce it to **E** such that **DE**=**AD**. Show that the △s **ABD** and **DEC** are congruent; arrange the sides and angles of the triangles in pairs of corresponding parts.

11 In Ex. 10, show also that the triangles **ACD** and **DEB** are congruent; arrange the sides and angles in pairs of corresponding parts.

12. **A** and **B** are two points on the level ground, between which lies an obstacle, (e.g. an elevated piece of ground) which prevents a straight line being laid from **A** to **B**. How will you find the distance between **A**, **B**?

Drive pegs at **A** and B, and a third peg at a point **C** such that the straight lines **AC**, **BC** are clear of the obstacle. Stretch a string from **A** to **C** and extend it beyond **C** (in a straight line with **AC**) to a point **D** such that **CD**=**AC**. Similarly stretch a

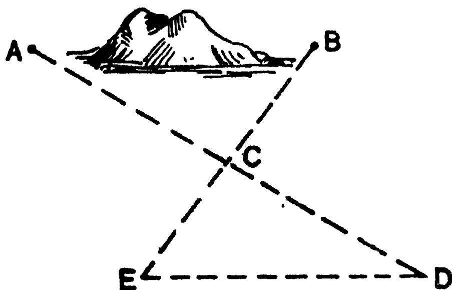


Fig. 129

string from **B** to **C** and extend it in a straight line **BC** to a point **E** such that **CE**=**BC**. Drive pegs at **D** and **E**, and measure the distance **DE** with a tape; the distance **DE**=the distance **AB**. Prove the result.

13. **O** is the centre of a circle; **OA**, **OB**, **OC** are three radii, **OB** lying between **OA** and **OC**. If the ∠**AOB**=∠**COB**, show that **OB** bisects the chord **AC** at right angles.

THEOREM 5. [Euclid 1, 5]

57. If two sides of a triangle are equal, the angles opposite to these sides are equal.

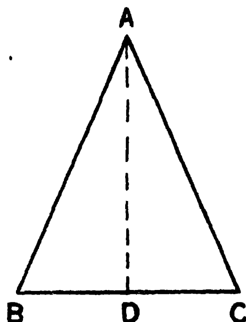


Fig. 130

The △ **ABC** has its sides **AC, AB** equal.

To prove that the angle opposite to **AC** = the angle opposite to **AB**, i. e., ∠**B** = ∠**C**.

Suppose **AD** is the bisector of the ∠**BAC**, so that ∠**BAD** = ∠**CAD**.

Proof :—In the △s **ABD, ACD**,

{ **AB**=**AC** ; (*given*)
the side **AD** is common to both the △s ;
the included ∠**BAD** = the included ∠**CAD**.

Hence the △s are congruent ; (*Theor.* 4)

∴ the ∠**ABC** = the ∠**ACB**.

Note 1. From the congruence of the △s **ABD, ACD** also follow .—

(i) **BD**=**CD**,

(ii) ∠**ADB**=∠**ADC** ; each of these ∠s is therefore a right angle. [Theor 1, Note 2].

Thus **the bisector of the ∠BAC bisects BC and is perpendicular to it.**

Note 2. If a triangle is equilateral it is also equiangular. This follows at once from theorem 5, and may be regarded as a *corollary* to this theorem.

A corollary to a theorem is a result which follows readily from that theorem.

Note 3. Theorem 5 may also be proved in either of the following ways.

(i) Fold the triangle **ABC** (Fig. 130) through **A** so that **AC** falls along **AB**, then since **AC = AB**, **C** will fall on **B**; and if the crease cuts **BC** at **D**, **CD** will fall upon and coincide with **BD**. Thus the ∠**ACD** fits the ∠**ABD** exactly and the two angles are therefore equal. Note that *the crease bisects the* ∠**BAC** as well as the side **BC**.

(ii) Since **AB = AC**, there is no reason why the angle opposite to one of them should be greater than the angle opposite to the other; any reason for supposing one of the angles to be the greater could equally be applied to the other. Such a proof may be called **'Proof of sufficient reason.'**

Note 4. The bisector of teh ∠**BAC** is such that if the triangle were folded along this line, [See Note 3 (i)] the part **ACD** would fall upon and coincide with the part **ABD**.

When a straight line is so situated with respect to a figure, that a folding along this line would bring into coincidence the two parts of the figure lying on the two sides of the line, the line is called an **Axis of Symmetry**; and the figure is said to be **symmetrical about the line.**

Thus an isosceles triangle is symmetrical about the bisector of the angle between the equal sides. An equilateral triangle is symmetrical about the bisectors of its angles: it has thus three axes of symmetry.

A circle is symmetrical about a diameter (see § 30). How many axes of symmetry has a circle got ?

Have a pair of intersecting lines an axis of symmetry ? If so, how many ? What are they ?

Note 5. The equal sides of an isosceles triangle are often referred to as **'the sides'**; the third side is then called **'the base'**, the point at which the equal sides meet is called **'the vertex'**, and the angle at this point i.e. the angle between the equal sides is called the **'Vertical angle.'** Thus Theorem 5 might be stated in the following way :—**'The angles at the base of an isosceles triangle are equal.'**

EXERCISE XVII.

1. Construct a number of isosceles triangles, and verify in each case by actual measurement, that the angles opposite to the equal sides are equal.

2. The base **BC** of an isosceles triangle **ABC** is produced both ways Show that the exterior angles so formed are equal.

3. The equal sides **AB, AC** of an isosceles triangle **ABC** are produced to **D** and **E**; show that the exterior angles **CBD** and **BCE** are equal.

4. **ABCD** is a quadrilateral with all its sides equal; **AC** and **BD** are joined. Show that (i) $\angle$**ACB** = $\angle$**CAB**, (ii) $\angle$**ACD** = $\angle$**CAD**, (iii) $\angle$**ABD** = $\angle$**ADB**, (iv) $\angle$**CBD** = $\angle$**CDB**.

5. **ABCD** is a quadrilateral with all its sides equal. Show that $\angle$**A** = $\angle$**C** and $\angle$**B** = $\angle$**D**.

6. On the same base **BC** but on opposite sides of it stand two isosceles triangles **ABC**, and **DBC**. Show that $\angle$**ABD** = $\angle$**ACD**.

7 On the same base **BC** and on the same side of it stand two isosceles triangles **ABC** and **DBC**. So that $\angle$**ABD** = $\angle$**ACD**

8 In each of Exs. 6 and 7, show that the $\triangle$s **ABD** and **ACD** are congruent, and deduce therefrom that **AD** bisects each of the $\angle$s **BAC** and **BDC**.

9. In each of the Ex. 6, 7 show that **AD** bisects **BC** at right angles.

10. **O** is the mid-point of a line **PQ**. From **O** a line **OR** is drawn equal to **OP** or **OQ**. Show that the Δ**PQR** has one of its angles equal to the sum of the other two.

11. On the base **BC** of an isosceles triangle **ABC** two points **P** and **Q** are taken such that **BP = CQ,** show that **AP = AQ**, and $\angle$**APQ =** $\angle$ **AQP**.

12. The mid-point **O** of the base of an isosceles triangle is joined to the mid-points **P** and **Q** of the equal sides, show that **OP = OQ**.

13. Show that the triangle formed by joining the mid-points of the sides of an equilateral triangle is also equilateral.

14. Two circles of centres **O** and **O′** intersect in the points **A** and **B**. Show that the common chord **AB** is bisected at right angles by the line joining the centres i. e. **OO′**. [See Ex. 9].

THEOREM 6. [Euclid 1, 6].

58. **If two angles of a triangle are equal, the sides opposite to these angles are equal.** *(The converse of Theorem 5).*

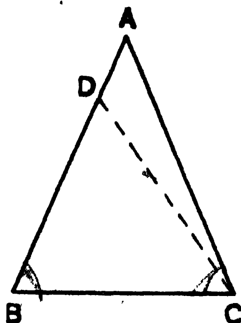


Fig. 131

The △ **ABC** has ∠**B** = ∠**C**.

To prove that **AC** = **AB**.

Proof :—If **AC** and **AB** are not equal, suppose **AB** is greater than **AC**. From **AB** cut off **BD** equal to **AC**. Join **CD**.

In the △ˢ **ACB, DBC**

{ **AC** = **BD**,
BC is common,
the included ∠**ACB** = the included ∠**DBC**, (*given*)

∴ the △ˢ are congruent, and therefore (*Theor.* 4)
equal in area (Read remarks § 55)

But this is impossible because the △ **DBC** is a part of the △ **ACB**.

Hence **AB** and **AC** cannot be unequal,
that is, **AC** = **AB**.

Corollary. *If a triangle is equiangular it is also equilateral.*

Note 1. The proof given above is called an **indirect proof;** it proves the truth in an indirect way by showing that the denial of the truth leads to an absurd result Thus by denying the equality of the sides **AB** and **AC** we are led to an absurd result, viz., a whole is equal to a part.

The Latin name for such a proof is proof by **'reductio ad absurdum'**—(*proof leading to the impossible*).

A *direct* proof[1] of Theorem 6 may be devised in the following way.

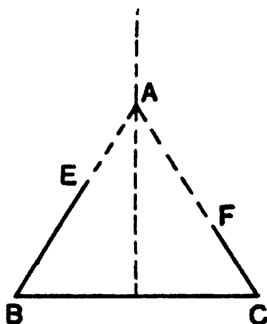


Fig. 132

The angles **EBC** and **FCB** (Fig. 132) at the ends of the line **BC** are equal. If the figure is folded such that **C** falls on **B**, the crease (shown by the dotted line) will divide **BC** into two equal parts, and these parts will come into coincidence after folding. And because ∠**B**=∠**C**, **CF** will fall along **BE**. If the figure were now opened out and **BE** and **CF** were produced to meet the crease, they would meet the crease in the same point **A**, otherwise **CF** would not fall along **E** after folding (see Fig. 133). It is now readily seen how the vertex **A** of the △ formed by the line **BC**, **BE**, **CF** (Fig. 132) lies on the crease and how the same folding which brings the point **C** to lie upon the point **B** will make the side **AC** fall upon and coincide exactly with the side **AB**, thus **AC**= **AB.**

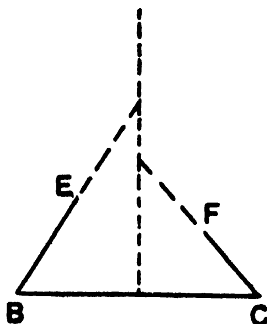


Fig. 133

Can you not devise a proof of Theorem 6 by the principle of *sufficient reason* as in the case of Theorem 5. [see Theor. 5 Note 3, (ii)]

1. See D. Mair's *School Course of Mathematics*, § 10 (Clarendon Press, Oxford).

EXERCISE XVIII.

1. Draw five different triangles, each with two of its angles equal and verify in each case (by measurement) that the sides opposite to the equal angles are equal.

2. **ABC** is a triangle, of which the sides **AB** and **AC** are produced to **P** and **Q** respectively. Show that if $\angle$**CBP** = $\angle$**BCQ**, the $\triangle$ **ABC** is isosceles.

3. The bisectors of the base angles **B** and **C** of an isosceles triangle **ABC**, meet at **E**; show that the triangle **EBC** is isosceles.

4. The external bisectors of the base angles **B** and **C** of an isosceles triangle **ABC** meet in **O**, show that the $\triangle$ **OBC** is isosceles.

5. **ABCD** is a quadrilateral such that $\angle$**B** = $\angle$**C** and **AB** = **CD**. If **AB** and **CD** meet at **O** when produced, show that **OA** = **OD**.

6. **ABCD** is a quadrilateral such that $\angle$**B** = $\angle$**C**, and **AB** = **CD**. If **AC** and **BD** meet at **P**, show that the $\triangle$s **BPC** and **APD** are isosceles.

7. In Ex. 3, show that **EA** bisects the angle **BAC**.

8. In a quadrilateral **PQRS**, if **PQ** = **QR** and $\angle$**P** = $\angle$**R**, show that **PS** = **SR**.

THEOREM 7. [Euclid 1, 8].

59. If two triangles have three sides of the one equal to three sides of the other, each to each, the triangles are congruent.

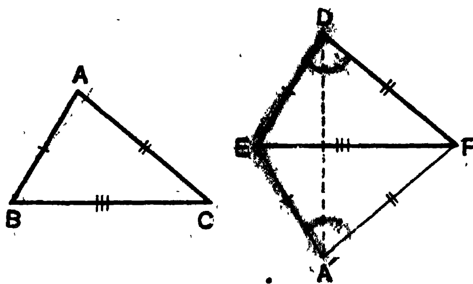


Fig 134

△s **ABC, DEF** have
AB = **DE**,
BC = **EF**,
and **CA** = **FD**,

To prove that the △s are congruent.

Proof :—Suppose the △ **ABC** is taken up and placed with the side **BC** exactly fitting the equal side **EF** of the △ **DEF**, **B** falling on **E**, and **C** on **F**, and the point **A** falling at a point **A′** on the side of **EF** opposite to **D**. Join **DA′**.

Because **EA′** = **AB**, and **ED** = **AB**,

∴ **EA′** = **ED**,

∴ ∠ **EA′D** = ∠ **EDA′**. (*Theor.* 5)

Similarly $A'F = DF$,

$\therefore \angle FA'D = \angle FDA'$.

Hence the whole $\angle EA'F$ = the whole $\angle EDF$.

But the $\angle EA'F$ = the $\angle BAC$.

$\therefore$ the $\angle BAC$ = the $\angle EDF$.

Now the $\triangle^s$ ABC, DEF have

$AB = DE$,

$AC = DF$,

and the included $\angle BAC$ = the included $\angle EDF$;

therefore **the $\triangle^s$ ABC, DEF are congruent.** (*Theor.* 4.)

Note 1. From the fact of the three sides **AB**, **BC**, **CA** of the $\triangle$**ABC** being equal to the three sides **DE**, **EF**, **FD** of the $\triangle$**DEF**, is proved the perfect equality of the two $\triangle^s$, so that $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$. Note that the angles which are opposite to a pair of equal sides in the two triangles, are equal.

You can construct any number of triangles with sides of the same specified lengths (See Ex. 1, p 81). Theorem 7 shows that all these triangles will be congruent. Thus when the lengths of the three sides are assigned, the triangle is uniquely determined in size and shape.

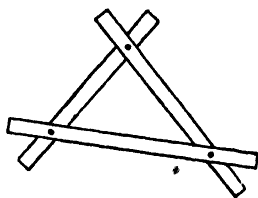


Fig. 135

Consider the triangle formed by three strips of wood hinged together with smooth pins, as shown in Fig. 135. Can the shape of the triangle be altered without breaking the joints ?

Note 2. In Fig. 134, **A'D** cuts **EF** at some point between **E** and **F**. But it may happen that **A'D** passes *through* **E** or **F** as shown in Fig. 136, or that **A'D** passes *outside* **EF** as shown in Fig. 137. That the proof will also hold in *all* such cases, may easily be seen by the student.

(a) *When* **A'D** *passes through* **F** (Fig. 136); **A'FD** is one straight line, and the △ **A'ED** is isosceles with **EA'=ED**,

∴ ∠**EDA'**=∠**EA'D**=∠**BAC**;

hence, by Theorem 4, △s **ABC, DEF** are congruent.

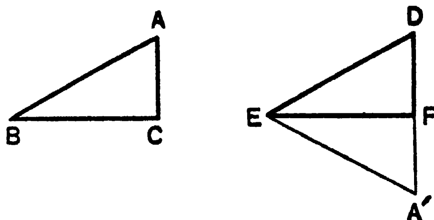


Fig. 136

(b) *When* **A'D** *passes outside* **EF** (Fig. 137) :—

△s **A'ED**, and **A'FD** are isosceles,

∴ ∠**EDA'**=∠**EA'D**

and ∠**FDA'**=∠**FA'D**,

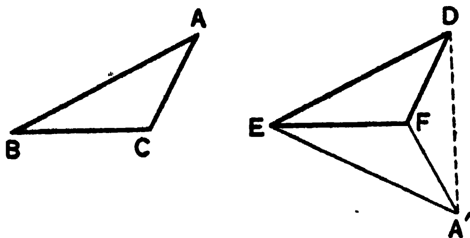


Fig. 137

hence the difference of the ∠s **EDA', FDA'**

=the difference of the ∠s **EA'D, FA'D**.

that is, ∠**EDF**=∠**EA'F**=∠**BAC**,

hence, by Theorem 4, △s **ABC, DEF** are congruent.

EXERCISE XIX.

1. **ABCD** is a quadrilateral such that **AB=AD**, and **CB=CD**. Show that **AC** bisects the ∠s **BAD** and **BCD**.

2. **ABCD** is a quadrilateral with all its sides equal. Show that **AC** bisects the ∠s **A** and **C**, and **BD**, ∠s **B**, **D**. Show also that **AC**, **BD** bisect each other at right angles.

3. Show that the line joining the vertex to the mid-point of the base of an isosceles triangle, bisects the vertical angle, and is perpendicular to the base

4. **ABC**, **DBC** are two isosceles triangles standing on the same base **BC**, and on the same side of it. Show that **AD** bisects the angles **BAC** and **BDC**. Show also that **AD** (or **DA**) produced bisects **BC** at right angles.

5. **AB**, **CD** are two equal chords of a circle whose centre is **O**, show that the ∠s **AOB**, **COD** are equal.

6. Show that the line joining the centre of a circle to the mid-point of any chord is perpendicular to the chord.

7. If the opposite sides of a quadrilateral are equal, the opposite angles will be equal.

8. In two circles, centres **O** and **O′** cut in **P** and **Q**, show that the ∠s **OPO′** and **OQO′** are equal.

PROBLEMS.

60. We shall now pass on to certain *problems* (e.g., the bisection of an angle, the bisection of a straight line, construction of perpendiculars to a given straight line, making an angle equal to a given angle) as affording good applications of the foregoing theorems. It should be noted that in the problems to be considered now, *the constructions are to be restricted to the use of ruler and compasses only* [see § 50].

PROBLEM 1.

To bisect a given angle.

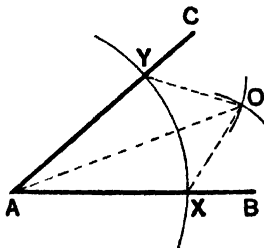


Fig. 138

Let **BAC** be the angle which is to be bisected (Fig. 138).

Construction. With centre **A** and any radius, draw an arc of a circle cutting **AB**, **AC** at **X** and **Y**.

With centres X and Y and any suitable[1] radius draw two arcs of *equal* circles cutting at O. Join AO.

Then AO bisects the ∠BAC.

Proof. Join **OX, OY.**

In the △s **AOX, AOY**

$\begin{cases} AX = AY \text{ (being radii of a circle)}, \\ OX = OY \text{ (being radii of equal circles)} \\ AO \text{ is common}, \end{cases}$

∴ the △s are congruent; (*Theor.* 7)

∴ ∠**OAX** = ∠**OAY**,

i.e. **AO bisects ∠BAC.**

Note. The equal circles drawn with centres **X** and **Y** in the above construction will cut each other in another point **O′**. (See Fig. 139) distinct from **O**. Will this point joined to **A** give the bisector of the ∠**BAC** ? Discuss (by drawing figures) the following cases :—

(i) When **O′** falls within the ∠**BAC**.

(ii) When **O′** falls at **A**.

(iii) When **O′** falls outside the ∠**BAC**.

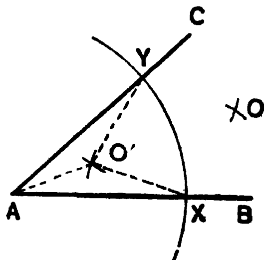


Fig. 139

EXERCISE XX.

1. Make an acute angle and bisect it.
2. Make an obtuse angle and bisect it.
3. How would you bisect a reflex angle ?
4. Make an angle 120°, and bisect it (i) with the aid of a protractor, (ii) by construction as in Problem I, [Test the accuracy of your protractor].

1. The radius must be chosen large enough so that the circles intersect. If the radius be too small the circles will not intersect as you may actually see by trial.

5. In what other way could you bisect an angle ?
[by folding, see § 27].

6. Of all the various methods of bisection which would you consider the most accurate, and which the most expeditious ?

7. Make an obtuse angle and divide it into four equal parts with ruler and compasses only

8. Draw a triangle **ABC**, such that **AB** = 2 in., **BC** = 3 in., **AC** = 2·5 in. Bisect the ∠**A** and measure the lengths of the parts into which **BC** is divided by the bisector of the ∠**A**. If **BD**, **DC** be these parts, do you find **BD**, **DC** are in the same ratio as **AB**, **AC** ?

9. Draw a triangle **ABC**, of good size and draw the bisectors of its angles. Do you find these bisectors meeting at a point ?

10. Draw a triangle **ABC**, produce **AB**, **AC** to **X** and **Y** respectively. Draw the bisectors of the ∠s **BAC**, **XBC**, **YCB**. Do you find these bisectors meeting at a point ?

PROBLEM 2.

61. To bisect a given straight line.

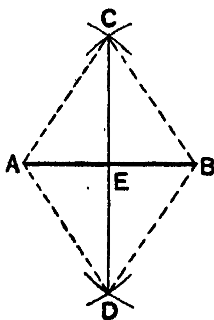


Fig. 140

Let it be required to bisect the straight line **AB** (Fig. 140)

Construction. With centres **A, B** and any suitable radius describe two equal circles intersecting at **C** and **D.** Join **CD** and let it meet **AB** at **E.**

Then **E** is the mid-point of **AB.**

Proof. Join **AC, BC, AD, BD.**

In the △ˢ **ACD, BCD,**

$$\begin{cases} AC = BC, & \text{(being radii of equal circles),} \\ AD = BD, & \text{,, ,, ,, ,,} \\ CD \text{ is common,} & \end{cases}$$

∴ the △ˢ are congruent; (*Theor.* 7)

∴ ∠**ACD** = ∠**BCD.**

Now in the △ˢ **ACE, BCE,**

$$\begin{cases} AC = BC, & \\ EC \text{ is common,} & \\ \angle ACE = \angle BCE & \text{(Proved)} \end{cases}$$

∴ the △ˢ are congruent,

∴ **AE** = **BE,**

i.e., **AB is bisected at E.**

Note. From the congruence of the △ˢ **ACE, BCE** it also follows that ∠**AEC** = ∠**BEC**, hence each of these angles is a right angle. [See Theorem 1 (§ 52) note 2]. Thus **CD** not only bisects **AB** but is also perpendicular to it

The **perpendicular bisector** of a given straight line is that straight line which bisects it at right angles.

Thus the construction in Problem 2 gives us the perpendicular bisector of **AB**.

---

1. The whole circles need not be drawn, only *arcs*, showing the points of intersection, are shown.

EXERCISE XXI.

1. Draw three straight lines and bisect each of them (with ruler and compasses only).

2. Draw a straight line and quadrisect it.

3. Draw a triangle **ABC** so that **AB** = 2 in., **BC** = 2·5 in., and **CA** = 3·2 in. Find the mid-points, **D**, **E**, **F** of **BC**, **CA**, **AB** respectively. Join **AD**, **BE**, **CF** and measure them. Do you find **AD**, **BE**, **CF** concurrent?

4. Draw a straight line and construct the perpendicular bisector.

5. Draw a triangle, and construct the perpendicular bisectors of its sides. Do you find these bisectors concurrent?

6. State the various ways in which you might bisect a line. [See Exs. 10, 11, 12 p. 24].

PROBLEM 3.

62. **To draw a straight line perpendicular to a given straight line from a given point in it.**

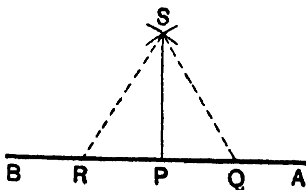


Fig. 141

Let it be required to draw a perpendicular to AB from the point P in it (Fig. 141).

Construction. Cut off any equal lengths[1] **PQ, PR** from the line **AB.**

With centres **Q** and **R,** and any convenient[2] radius, describe equal circles intersecting at **S.** Join **PS.** Then **PS** is perpendicular to **AB.**

Proof. In the △s **QPS** and **RPS,**

{ **QS = RS.** (being radii of equal circles),
PQ = PR.
SP is common ;

∴ the △s are congruent. (*Theor.* 7)

∴ the ∠**QPS** = ∠**RPS,**

∴ each of them is a right angle (see Theorem 1. Note 2),

that is, **PS is perpendicular to AB.**

Note. If the point **P** happens to be at (or very near) an extremity of the line, say, **A,** you will produce the line in the direction **BA,** and then proceed with the construction.

EXERCISE XXII.

(Protractor and set-squares must not be used in drawing the perpendiculars in the following examples.)

1. Draw a straight line **AB,** 7 cm. long. Mark points **C** and **D** in it such that **AC**=3 cm. and **AD**=6 cm. Draw two perpendiculars to **AB,** one from the point **C,** the other from the point **D.**

2. Draw any straight line **AB,** and at its extremities draw two straight lines at right angles to **AB.** From these lines cut off equal lengths **AD** and **BC.** Join **CD,** and measure the ∠s **ADC,** and **BCD** with your protractor.

---

1. This you may do with your compasses, by tracing a circle of any radius with **P** as centre.

2. See foot-note page 125.

3. Construct an angle of 45° without using a protractor. [Take any line **AB** and from any point **C** in it draw a line **CD** perpendicular to **AB**. Then bisect the ∠**ACD** (Prob. 1), and you will get an angle of 45°.]

4. Construct an angle of $22\frac{1}{2}°$ without using a protractor.

5. Take two lines **AOB, COD** intersecting at **O**. Draw the bisector **OX** of the ∠**AOC**, and from **O** draw **OY** perpendicular to **OX**. Prove that **OY** (or **YO** produced) bisects the ∠**BOC**. Check by measurement.

PROBLEM 4.

63. To draw a straight line perpendicular to a given straight line from a given point outside it.

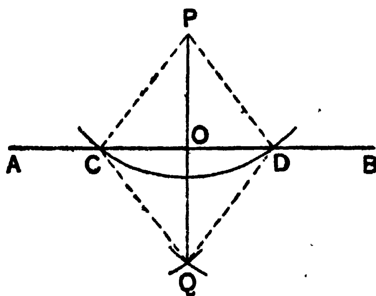


Fig. 142

Let it be required to draw a perpendicular to the line **AB** from the point **P** *outside* it.

Construction. With centre **P** and any convenient[1] radius draw a circle cutting **AB** in **C** and **D**.

With centres **C** and **D**, and any convenient radius, draw equal circles intersecting at **Q**. Join **PQ** cutting **AB** at **O**.

Then **PO** is ⊥ to **AB**.

Proof. Join **PC, PD, QC, QD.**

In the △s **PQC, PQD,**

PC=PD (being radii of a circle),
QC=QD (being radii of equal circles),
PQ is common ;

∴ the △s are congruent. (*Theor.* 7)

∴ ∠CPQ = ∠DPQ.

Now in the △s **POC, POD.**

PC = PD,
PO is common,
∠CPO = ∠DPO ; *Proved.*

∴ the △s are congruent. (*Theor.* 4)

∴ ∠POC = ∠POD,

∴ each of these angles is a right angle,

∴ PO is ⊥ to AB.

Note. One or both of the points **C** and **D** in which the circle with centre **P** cuts the line **AB** may not lie *between* the points **A, B.** In such a case the line **AB** must be produced one way, or both ways, as the case may be (see Fig. 143).

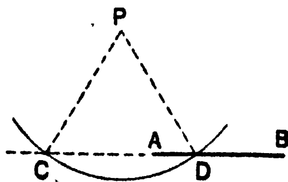


Fig. 143

1. See foot-note page 125.

EXERCISE XXIII.

1. Draw a line **AB** and mark three points **P, Q, R** as shown in Fig. 144 Draw perpendiculars from **P, Q, R** upon **AB**.

2. Draw a triangle **ABC** of good size and draw the perpendiculars from **A, B, C** upon **BC, CA, AB** respectively. *Do you find these perpendiculars concurrent?*



Fig. 144

3. Draw a triangle of sides 2 5 in, 3 in., and 3 5 in, and draw perpendiculars from the vertices upon the opposite sides

4. Make an acute angle **AOB**. Draw the bisector of this angle and take any point **P** in it. From **P** draw perpendiculars upon **OA, OB**. Do you find these perpendiculars equal?

5. Draw any circle and place a chord in it. Drop a perpendicular from the centre upon the chord

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PROBLEM 5.

64. At a given point in a given straight line to make an angle[1] equal to a given angle.

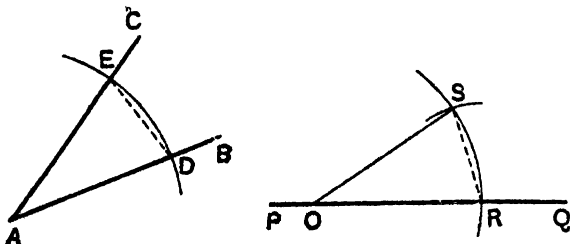


Fig. 145

1. With the given line for one of its arms.

At the point **O** in the line **PQ**, to make an angle equal to the angle **BAC**.

Construction. With centre **A** and any radius draw a circle cutting **AB**, **AC** in **D** and **E** respectively.

With centre **O** and the same radius draw a circle cutting **PQ** in **R**. Join **DE**.

With centre **R** and radius = **DE**, draw a circle cutting the former circle in **S**.

Join **OS**.

Then the ∠**ROS** = the ∠**BAC**.

Proof. Join **RS**.

In the △ˢ **ROS**, **DAE**.

{ **OR** = **AD** (being radii of equal circles),
OS = **AE** (being radii of equal circles),
SR = **DE**, (*Constr.*)

∴ the △ˢ are congruent. (*Theor.* 7)

∴ ∠**ROS** = ∠**DAE**, (i.e ∠**BAC**)

EXERCISE XXIV.

1. Make an obtuse angle **BAC** and draw a line **PQ**, and mark a point **O** in it. From **O** draw two lines **OR**, **OS** on opposite sides of **PQ** so that the ∠ˢ **ROP** and **SOP** are each equal to ∠**BAC**.

2. Take a line **AB** and from **A** draw another line **AC**; from **B** draw a line **BD** on the same side of **AB** as **AC**, such that (i) the angle **ABD** = ∠**BAC**. (ii) the angle between **BD** and **AB** *produced* = ∠**BAC**.

3. Draw a triangle **ABC**, and take any line **PQ**. On **PQ** construct a triangle **PQR** such that ∠**P** = ∠**A** and ∠**Q** = ∠**B**.

CHAPTER IV.

ON INEQUALITIES IN TRIANGLES.

THEOREM 8. [Euclid I. 16]

65. **If one side of a triangle is produced, the exterior angle is greater than either of the interior opposite angles.**[1]

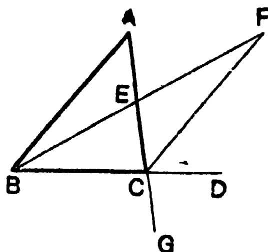


Fig. 146

ABC is a triangle of which the side **BC** is produced to **D**.

To prove that the exterior angle **ACD** is greater than either of the *interior opposite* ∠ˢ **ABC, BAC.**

Suppose **E** is the mid-point of **AC**. Join **BE**, and produce it to **F** making **EF = BE.** Join **CF.**

Proof. In the △ˢ **AEB, CEF,**

$$\begin{cases} AE = CE, \\ BE = FE, \\ \angle AEB = \text{vert. opp. } \angle CEF; \end{cases}$$

∴ △ˢ **AEB, CEF** are congruent.

∴ **∠BAE = ∠ECF.**

---

1. In any triangle **ABC** (Fig. 146), ∠ˢ **BAC, CBA, ACB** may be called *interior* angles (being all within the △). If a side, say **BC**, is produced to **D**, then ∠**ACD** is an *exterior* angle, and of the three interior angles, ∠**ACB** is adjacent to ∠**ACD** and the other two angles, viz., **BAC** and **CBA** are each said to be opposite to ∠**ACD**.

It will be proved later that an exterior angle is equal to the sum of the two interior opposite angles.

But ∠ ACD is greater than its part, ∠ ECF.

∴ ∠ ACD is greater than ∠ BAE (i.e. ∠ BAC).

Again by producing AC to G, we may prove in a similar way[1] that ∠ BCG is greater than ∠ ABC.

But ∠ BCG = the vert. opp. ∠ ACD,

∴ ∠ ACD is greater than ∠ ABC.

It is thus proved that ∠ ACD is greater than either of the ∠ˢ BAC, ABC.

Corollary 1. **(Any two angles of a triangle are together less than two right angles.)**

For example, ∠ A + ∠ C (Fig. 147) is less than 2 right angles.

Produce AC to D, then

∠ A is less than the ext. ∠ BCD.

∴ ∠ A + ∠ C is less than ∠ BCD + ∠ ACB, that is, less than 2 right angles. [See Theor. 1].

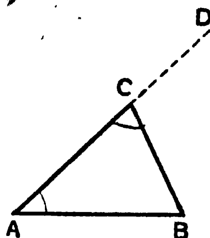


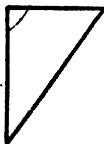
Fig. 147

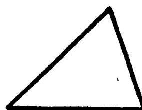
From the above follows :

A triangle cannot have more than one of its angles obtuse or right.

In other words, a triangle must have *at least* two of its angles acute.


Fig. 148


Fig. 149


Fig. 150

Def. If a triangle has one of its angles obtuse, it is called an **obtuse-angled** triangle (Fig. 148) ; if one of the angles is a right angle,

1. By joining **A** to the mid-point of **BC** and proceeding as before.

2. It will be proved later that the *three angles of a triangle are together equal to two right angles.* (See § 46).

it is called a **right-angled** triangle (Fig. 149). An **acute-angled** triangle is one which has all its angles acute (Fig. 150.)

In a right-angled triangle, the side opposite to the right angle is called the **Hypotenuse** and the other two sides are then referred to as '**the sides**'.

Corollary 2. From an outside point only one perpendicular can be drawn to a given straight line.

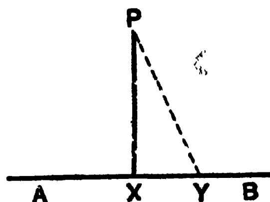


Fig. 151

For if there were two perpendiculars **PX, PY** from **P** upon **AB**, we would have a triangle **PXY** with two of its angles *right*, which is impossible.

The student should also note that only one perpendicular can be drawn to a line from a given point in the line.

EXERCISE XXV.

1. If a side of a triangle be produced both ways, show that the sum of the exterior angles so formed is greater than two right angles.

2. **ABC** is a triangle; **BC** is produced to **D**, **CA** to **E**, and **AB** to **F**. Show that any two of the ∠s **DCA**, **EAB**, **FBC** are together greater than two right angles.

3. **ABC** is a triangle and **O** is any point within it; show that ∠**BOC** > ∠**BAC**, ∠**COA** > ∠**CBA**, and ∠**AOB** > ∠**ACB**.

If **O** be on the side **BC**, will the above inequalities hold ?

4. Show that the equal angles of an isosceles triangle are each acute

5. If the angles **B** and **C** of a triangle **ABC** are both acute, show that the foot of the perpendicular dropped from **A** upon **BC** must lie *between* the points **B** and **C**.

6. If the ∠**B** of a △ **ABC** be obtuse, show that the foot of the perpendicular from **A** upon **BC** will lie on **CB** produced, and the foot of the perpendicular from **C** upon **AB** will lie on **AB** produced.

7. With a point **O** as centre, a circle is drawn cutting a given straight line **PQ** (which does not pass through **O**) in the points **A** and **B**; then **OA**=**OB**, being radii of the circle. Can any *other* line **OC** be drawn from **O** to the line **PQ**, which shall be of the same length as **OA** and **OB**?

THEOREM 9. [Euclid I. 18.]

66. If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.

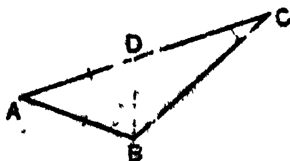


Fig. 152

ABC is a triangle in which AC > AB.

To prove that ∠ABC > ∠ACB.

From AC, the greater side, cut off AD = AB.

Join BD.

Proof. In the △ABD, because AB = AD,

∴ ∠ABD = ∠ADB. (*Theor.* 5)

But in the △ CBD,

the ext. ∠ADB > the int. opp. ∠DCB. (*Theor.* 8)

∴ ∠ABD > ∠DCB.

But ∠ABC > ∠ABD.

∴ ∠ABC > ∠DCB, i.e. ∠ACB.

EXERCISE XXVI.

1. In a scalene triangle, the angles are unequal

2. In a triangle, the angle opposite to the greatest side is the greatest.

3. In a triangle **ABC**, **AB** = 5 cm., **BC** = 4 cm., **CA** = 7 cm. Which angle is the greatest ? And which is the smallest ? Construct the triangle and verify by measurement.

4. If one side of a triangle is less than another, the angle opposite to the lesser side must be acute.

5. The angles adjacent to the greatest side of a triangle are both acute.

6. **ABCD** is a quadrilateral in which **AB** is the shortest and **CD** the longest side. Show that $\angle$**A** > $\angle$ **C**, and $\angle$**B** > $\angle$**D**.

THEOREM 10. [Euclid 1. 19].

67. **If two angels of a triangle are unequal, the greater angle has the greater side opposite to it.**

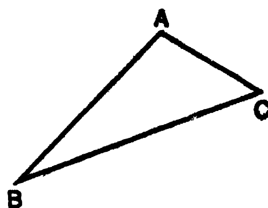


Fig. 153

ABC is a triangle in which ∠**C** > ∠**B**.

To prove that, **AB** > **AC**.

Proof. If **AB** is not greater than **AC**, it must be either less than **AC**, or equal to **AC**. Now if **AB** *were less* than **AC**, then ∠**C** would be less than ∠**B** (by Theor. 9); and if **AB** *were equal* to **AC** then ∠**C** would be equal to ∠**B** (by Theor. 5). But it is given that ∠**C** is greater than ∠**B**.

Therefore **AB** is neither less than, nor equal to **AC**.

Hence **AB must be greater than AC.**

EXERCISE XXVII.

1. If the angles of a triangle are unequal, the sides also must be unequal.

2. In a triangle, the side opposite to the greatest angle is the greatest, and the side opposite to the smallest angle is the shortest.

3. **In a right-angled triangle the hypotenuse is the longest side.**

4. In an obtuse-angled triangle, the side opposite to the obtuse angle is the greatest.

5. **AB** is a line and **O** a point outside it. From **O** a perpendicular **ON** is drawn to **AB**. If **X** be any point on **AB**, distinct from **N**, show that **ON** < **OX**.

6. **ABC** is an isosceles triangle, and any point **D** on the base **BC** is joined to **A**. Show that **AD** is less than **AB** (or **AC**).

If however **D** is a point on the base *produced*, **AD** is greater than **AB** (or **AC**).

7. The bisectors of the ∠s **B** and **C** of a triangle **ABC** meet in **O**. Show that if **AB**>**AC**; then **OB** > **OC**.

8. **ABC** is a triangle; a perpendicular **AD** is dropped from **A** upon **BC**. Show that **AB** > **BD** and **AC** > **CD**. Deduce that **AB**+**AC** > **BC**. [See Theorem 11].

THEOREM 11. [Euclid 1. 20.]

68. **Any two sides of a triangle are together greater than the third.**

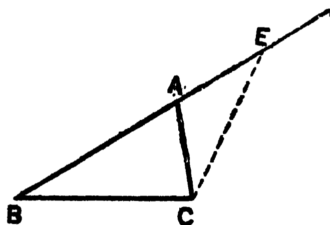


Fig. 154

ABC is a triangle,

To prove that

(i) BA + AC > BC,

(ii) CB + BA > CA,

(iii) AC + CB > AB.

Produce BA and cut off AE = AC.

Join EC.

Proof. In the △ AEC, AE = AC.

∴ ∠ACE = ∠AEC. (*Theor.* 5)

But ∠BCE > ∠ACE.

∴ ∠BCE > ∠AEC.

Now in the △ BEC, because, ∠BCE > ∠BEC,

∴ BE > BC. (*Theor.* 10)

But BE = BA + AE.

= BA + AC. (for AE = AC, constr.)

∴ BA + AC > BC.

Similarly it may be shown that CB + BA > CA.

and AC + CB > AB.

Note. Theorem 11 follows at once from the axiom, *the straight line joining two points is the shortest distance between them* (see § 41 Ex. 3).

EXERCISE XXVIII.

1. *Show that the difference of any two sides of a triangle is less than the third side.*

2. Show that any three sides of a quadrilateral are together greater than the fourth side.

3. Show that the perimeter of a quadrilateral is greater than the sum of its diagonals

4. Show that the diagonals of a quadrilateral are together greater than half its perimeter.

5. **O** is any point *within* a triangle **ABC**.
Show that (i) $OB+OC<AB+AC$.
(ii) $AB+BC+CA>OA+OB+OC$.
(iii) $OA+OB+OC > \frac{1}{2}(AB+BC+CA)$.

6. The sum of the distances of any point within a quadrilateral from the vertices is greater than half the perimeter of the quadrilateral.

7. Find the point the sum of whose distances from the vertices of a quadrilateral is the least. [The intersection of the diagonals.]

8. *The sum of any two sides of a triangle is greater than twice the median drawn through the vertex in which these sides meet.*

Note. The line joining a vertex of a triangle to the mid-point of the opposite side is called a **median** of the triangle. Thus if **ABC** be a triangle, and **D**, **E**, **F** are the mid-points of the sides **BC**, **CA**, **AE** respectively, then **AD**, **BE**, **CF** are the three medians of the $\triangle$ **ABC**.

[Let **ABC** be a triangle, and **D** the mid-point of **BC**; then **AB** + **AC** > 2**AD**. Produce **AD** to **E** so that **DE**=**AD**. Join **BE**. Then the △ˢ **ADC**, **BDE**, are congruent, so that **BE**=**AC**. From the △ **ABE**, **AB**+**BE** > **AE**.]

9. *In any triangle the sum of the medians is less than the perimeter.*

10. If the quadrilateral **ABC′D′** lies within the quadrilateral **ABCD** as shown on the margin (Fig. 155), show that **BC**+**CD**+**DA** > **BC′**+**C′D′**+**D′A**.

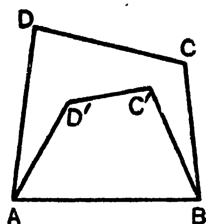


Fig. 155

THEOREM 12.

69. Of all straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest.

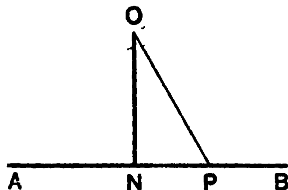


Fig. 156

ON is the perpendicular drawn to the line **AB** from the point **O** outside it ; let **OP** be any other straight line drawn from **O** to **AB**.

To prove that **ON** is less than **OP**.

Proof. In the △ **OPN**, since ∠ **ONP** is a right angle,

∴ ∠ **OPN** must be less than a right angle,

∴ ∠ **OPN** < ∠ **ONP**. [*Theor.* 8, cor. I.]

∴ **ON** < **OP**. (*Theor.* 10.)

Note 1. *If two obliques*[1] **OP**, **OQ** *are such that* **NP**=**NQ** where **ON** *is* ⊥ *to* **AB** (Fig. 157), *then* **OP**=**OQ**. This follows from the congruence of the △s **ONP**, **ONQ** (Theor. 4).

Note 2. *If two obliques* **OQ**, **OR** *are such that* **RN** > **QN** (Fig. 157), *then* **OR** > **OQ**. [∠**OQN** is acute,

∴ ∠**OQR** is obtuse ; also ∠**ORN** is acute,

∴ ∠**OQR** > ∠**ORQ**, hence **OR** > **OQ**]

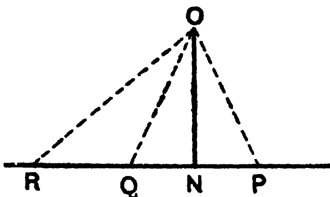


Fig. 157

1. By an *oblique* to a given straight line we mean a straight line which is drawn to the given straight line but which is not perpendicular to it.

The result holds also in the case when the obliques lie on opposite sides of the perpendicular **ON**: thus if **NR** > **NP**, it may be shown that **OR** > **OP**. The proof is left to the student.

Note 3. By the **distance of a point from a given line (AB)** we mean the *shortest distance* from the point to the line, it is thus given by the length of the perpendicular drawn from the point to the line.

EXERCISE XXIX.

1. Draw a triangle **ABC**, such that **AB**=5 cm, **BC**=6 cm., and **CA**=8 cm. Find the distances of the vertices from the opposite sides to the nearest millimetre by drawing the perpendiculars and measuring them.

2. Draw any line **AB**, and mark any point **P** in it. Through **P** draw a line perpendicular to **AB**, and on this line mark two points **P** and **Q**, (on opposite sides of **AB**), such that their distances from **AB** may be 3 cm.

3. Draw two lines **OX**, **OY** at right angles to each other. Mark a point **P** on **OX**, a point **Q** on **OY** and a point **R** not lying on **OX** or **OY**. State the distances of **P**, **Q**, **R** from **OX** and **OY** respectively.

CHAPTER V.

PARALLELS.

70. **Parallel straight lines.** Look at the two straight lines in which one of the walls of the class room meets the floor and ceiling. Do these lines meet? Would they meet if produced indefinitely in either direction? Consider similarly the opposite edges of a sheet of paper, of the top of a table, the two side-pieces of a ladder.

Draw a straight line **XY** (Fig. 158) and place a set-square[1] **ABC** with one of its edge **AC** lying along **XY**. Trace the line **AB** on the paper. Now slide the set-square along **XY** to a different position **A′B′C′** and trace the line **A′B′**. Remove the set-square and the lines appear as in Fig. 159. It should be observed that **AB, A′B′** *make equal angles with, and on the same side of,* the line **XY** (the angles being each equal to the angle **BAC** of the set-square)

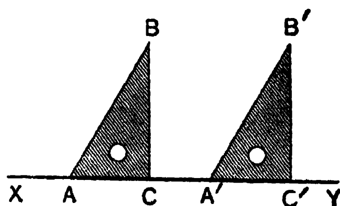


Fig. 158

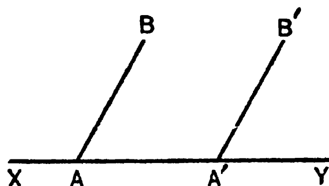


Fig. 159

1. A set-square is a triangular piece, one of the angles being a right angle. There are two varieties, one with angles 30°, 60°, and

Now look carefully at the lines **AB, A′B′**. Do they appear to be such that they would meet if produced long enough? Produce the lines **AB, A′B′** both ways (Fig. 160), and mark a number of points **P**, $\mathbf{P}_1$, $\mathbf{P}_2$, .., on **AB**. At these points erect perpendiculars to **AB**, meeting **A′B′** in **Q**, $\mathbf{Q}_1$, $\mathbf{Q}_2$......Measure the angles these perpendiculars (to **AB**) make with **A′B′** and also the length of each perpendicular. It will be found that any perpendicular to **AB** is also perpendicular to **A′B′**, and that the perpendicular distance between **AB** and **A′B′** is the same at all points.

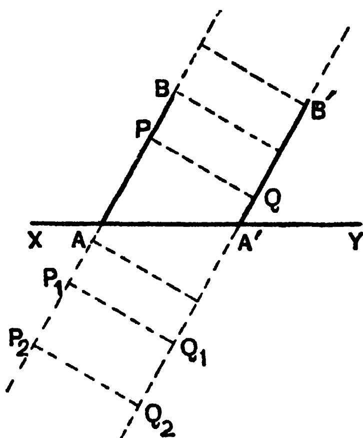
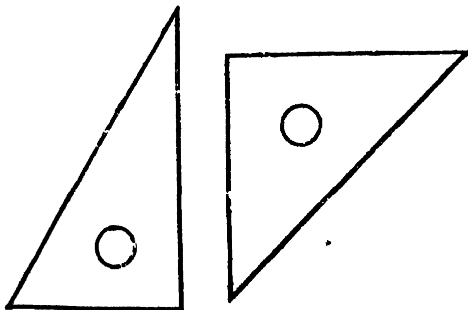


Fig. 160

90°, the other, with angles 45°, 45°, and 90°. Set-squares are generally made of such materials as pear-wood, celluloid, vulcanite, steel etc. They are used in tracing perpendiculars and parallels


Fig. 161—Set-squares.

It is thus experimentally verified that the lines **AB** and **A′B′** do not meet.

If two straight lines lie in a plane, and do not meet however far they may be produced (either way), they are said to be parallel.

Thus the lines **AB, A′B′** (Fig. 160) are parallel. This experimental result may be put in a more general form :

If at a number of points on any line **XY**, *straight lines,* **AB, A′B′, A″B″**, *are drawn making equal angles (of any size) with, and on the same side of,* **XY** (Fig. 162), *then these lines will be parallel to one another.*

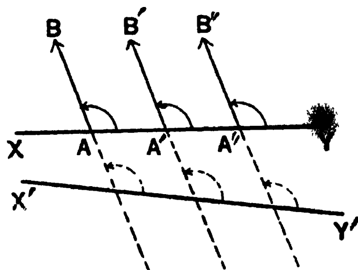


Fig. 162

Note. If you draw any other line **X′Y′** (Fig. 162) cutting the parallels, **AB, A′B′, A″B″**..., and measure the angles which the *directions* **AB, A′B′, A″B″**,... make with the *direction* **X′Y′** you will find these angles to be equal.

In particular, if the line **X′Y′**, were perpendicular to *one* of the lines **AB, A′B′, A″B″**,..., it would be perpendicular to each of them.

Thus *if two lines are parallel, a perpendicular to one of them is also perpendicular to the other.*

71. Construction of Parallels. Various mechanical devices, based on the results obtained above, may be used for drawing parallels.

(*a*) *With a ruler and a set-square (or a pair of set-squares).* Suppose it is required *to draw through the given* point *P, a line parallel to the given line AB* (Fig. 163).

Place a set square with one of its edges, **X**, lying along the given line **AB**; then place a ruler against a second edge, **Y**, of the set-square. Holding the ruler firmly, slide the set-square along it until the edge

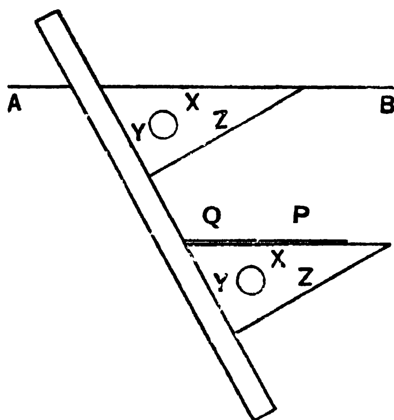


Fig. 163

X comes against the point **P**, through which the parallel is to be drawn. Trace the line **PQ** along the edge **X**, then **PQ** is parallel to **AB**. (A second set-square might be used instead of the ruler). By sliding the set-square to different positions and tracing a line along the edge **X** in each position, we may draw any number of parallels to the given line **AB**.

A set-square and a ruler (or a pair of set-squares) may also be used to draw through a given point **P**, *a perpendicular to a given straight line* **AB** (Fig. 164).

Place a set-square [position, marked (1)] with one of its *short* edges **X**, lying along the given line **AB**; then place a ruler against the longest edge

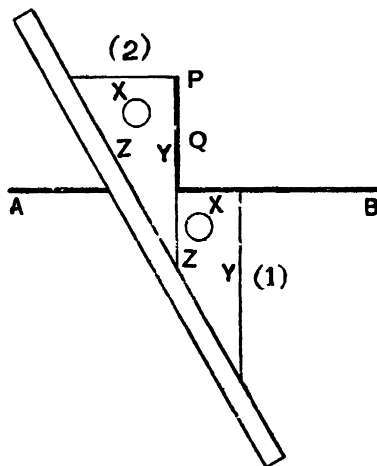


Fig. 164

Z (the hypotenuse) of the set-square. Holding the ruler firmly, slide the set-square along it, until the other short edge, **Y**, comes against the point **P** from which the perpendicular is to be drawn [position, marked (2)]. Trace the line **PQ** along the edge **Y**; then **PQ** is perpendicular to **AB**.

(*b*) *With a drawing-board and T-square*[1]. The T-square is placed on the board as shown in Fig. 165, with the stock S set against the left-hand edge of the board. By sliding the stock along the edge of the board, and tracing a line along the edge of the

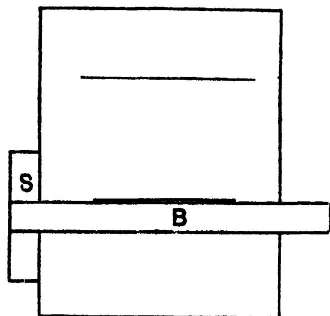


Fig. 165

blade B of the T-square in each position, we get a number of parallel lines.

By holding the T-square in a definite position and sliding a set-square along the edge of the blade we may draw another series of parallel lines.

A third device for drawing parallel lines consists in using a pair of *parallel rulers*; it will be explained later [see Ex. 22, (XXXVIII)].

Note. In the above constructions of parallel lines we have made use of certain instruments which are other than ruler and compasses (e.g. the set-square and the T-square). *Can parallel lines be constructed with ruler and compasses only?* Yes, the method will be explained later (see §§ 50, 90)

1. A T-square consists of a ruler called the *blade* (marked '**B**' in Fig. 165) which is firmly attached to a cross-piece called the stock (marked '**S**' in the figure). When the blade lies flat on the board, the lower face of the stock goes below the surface of the board.

72. Exercise on the construction of parallels. Parallelogram, Rectangle, Square, Trapezium.

EXERCISE XXX.

1. Make an $\angle XAY = 60°$ (Fig. 166). From **AX**, **AY** cut off **AB** = 1·5 in., and **AC** = 1 in. respectively.

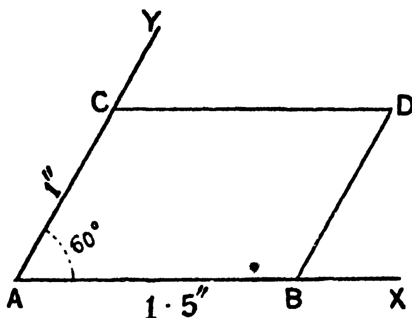


Fig. 166

Through **B** draw **BD** parallel to **AY**, and through **C** draw **CD** parallel to **AX**. The quadrilateral **ABDC** thus obtained, has its opposite sides parallel.

A quadrilateral with its opposite sides parallel is called a **Parallelogram.**

2. Construct a parallelogram **PQRS** such that $\angle P = 110°$, **PQ**=5 cm., **PS**=6 cm. Measure **QR**, **RS**. Do you find **QR**=**PS**, and **RS**=**PQ**. Measure $\angle$s **Q**, **R**, **S**. Do you find $\angle P = \angle R$ and $\angle Q = \angle S$?

3. Construct any parallelogram you like. Measure the sides and the angles. Do you find opposite sides equal, and opposite angles equal?

4. Construct (as in Ex. 3) three more parallelograms and measure the sides and angles. Do you find opposite sides and opposite angles equal in each case?

Can you say *that the opposite sides and angles of any parallelogram are equal* ?

5. Construct a parallelogram with two adjacent[1] sides measuring 2·5 in., and 2 in. respectively, and including an angle of 70°

6. Do Ex. 5 taking the included angle to be a right angle.

7. Draw a line **AC** and construct a parallelogram with **AC** for a diagonal. How many such parallelograms can you draw ?

8. Draw two lines **AB, AC.** Construct a parallelogram *with* **AC** *for one of its diagonals and* **AB** *for one of its sides.*

How many such parallelograms can you draw ?

9. Construct a parallelogram with a given line **AC** for a diagonal, and with one of its angle a right angle. How many such parallelograms can you have ?

Note. A parallelogram with one of its angles a right angle is called **a rectangle.**

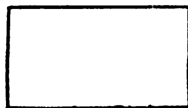


Fig. 167

10. Draw a rectangle **ABCD**, with **AB**=6 cm., and **AD**= 5 cm., and ∠**A**=a right angle. Measure the other three angles **B, C, D**. Do you find them to be right angles ?

11. Draw rectangles **ABCD** to the following measurements.

(i) **BC** = 2 in., **CD** = 2·2 in., ∠**C** = a right angle.
(ii) **AD** = 4 cm., **CD** = 5 cm., ∠**D** = a right angle.
(iii) **AB** = 3 cm., **BC** = 2·5 cm., ∠**B** = a right angle.
(iv) **AB** = 2 cm , **AD** = 1·5 cm., ∠**A** = a right angle.
(v) **AC** = 3 in., ∠**D** = a right angle.
(vi) **AC** = 2 in., ∠**A** = a right angle.
(vii) **BD** = 5 cm., ∠**A** = a right angle.

Measure the other three angles in each of the figures you draw. Do you find them to be right angles in each case ?

1. Next to each other.

Can you say generally that *all the angles of a rectangle are right angles ?*

12. In Ex 11, how many different rectangles can you draw in each of the cases (v), (vi), (vii) ?

13. Draw a rectangle **ABCD** such that **AB=AD=2** in. Measure **BC, CD.** Do you find them each to be 2 inches ?

14. Draw three different rectangles, each having a pair of adjacent sides equal. In each case measure the other two sides.

Can you say that *if a rectangle has a pair of adjacent sides equal, then all the four sides are equal* ?

Note. A rectangle with two adjacent sides equal is called a **square.** Can you say that all the sides of a square are equal, and all its angles are right angles ?

15. Draw a line **AB**=3 in., and mark a point **O** at a distance of 1 inch from **AB.** Through O draw a parallel to **AB.** Find two points **P** and **Q** on this parallel, such that **AP**=1·2 in., and **BQ**=1·6 in.

Evidently **APQB** is a quadrilateral with the opposite sides **AB, PQ** parallel. Consider the other pair of opposite sides **AP, BQ** ; are they parallel ?

If a quadrilateral has a pair of opposite sides parallel, but the other pair not, it is called a **trapezium.**

16. Draw a triangle **ABC** such that **AB**=4 cm., **BC**=5 cm., and **CA**=6 cm. Draw any line parallel to **BC**, and cut off from it a length **QR**=3 cm. Through **Q** draw **QP** parallel to **BA** and through **R** draw **RP** parallel to **CA.** Measure the angles of the △ **PQR**, do you find them equal to the angles of the △ **ABC** ?

17. In Ex. 16, measure the sides of the △ **PQR.** Calculate the ratios $\frac{PQ}{AB}$, $\frac{QR}{BC}$, $\frac{RP}{CA}$, Do you find them equal ?

18. Draw any other triangle **P′Q′R′** [see Ex. 16] with its sides **P′Q′, Q′R′, R′P′** respectively parallel to **AB, BC, CA** ; Measure the angles and the sides of the △ **P′Q′R′.** Do you find ∠**P′**, ∠**Q′**,

∠R′=∠A, ∠B, ∠C. respectively? Do you find the ratios $\frac{P'Q'}{AB}$, $\frac{Q'R'}{BC}$, $\frac{R'P'}{CA}$ equal?

19. Consider the rectangular block (Fig. 168). Point out the six rectangular faces which bound it. Do **AB** and **A′B′** lie in a plane? In which plane? Are they parallel? Will **AB** and **DD′** ever meet? Do they lie in a plane? Can you say **AB** and **DD′** are parallel? [**AB** and **DD′** never meet, *but they do not lie in a plane,* hence they are not parallel; two lines which do not meet are said to be **skew** *if they do not lie in a plane*]. Do **AB** and **C′D′** ever meet? Do they lie in a plane? Can you say that **AB**, **C′D′** are parallel? Are **AB**, **A′B′**, **CD**, **C′D′**, parallel to one another? Are **AA′**. **BB′**, **CC′**, **DD′** parallel? Are **AD**, **A′D′**, **BC**, **B′C′** parallel?

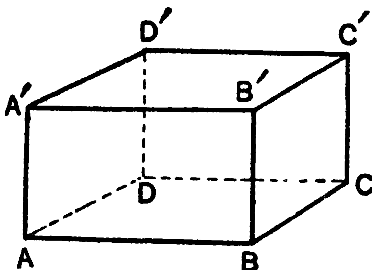


Fig. 168

Point out two pairs of skew straight lines.

73. **Parallel straight lines indicate same direction or opposite directions.** In Fig. 162, it will be seen that the directions AB, A′B′ A″B″ *differ from the direction* XY *by the same angle*; or, as we may also put it, the directions AB, A′B′ A″B″ may be obtained by turning the line XY *through the same angle in the same sense* in each case, round the points A, A′, A″ respectively. These considerations will naturally lead us to regard the parallel lines AB, A′B′, A″B″ (taken in the sense, from A to B, from A′ to B′, and from A″ to B″ respectively) *as indicating one and the same direction.* If a person were walking along AB and another along A′B′, or A″B″,

we would say that the two persons were walking in the same direction.

If, however, the line **BA** be taken *as pointing from* **B** to **A** and **A′B′** as *pointing from* **A′** *to* **B′** they would indicate *opposite* or *contrary directions.*

Thus a pair of parallel straight lines will indicate **same** or **opposite** directions according as they are taken in similar or dissimilar senses as shown in Fig. 169.

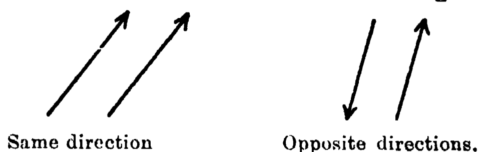


Fig. 169

74. **A second way of regarding parallels.**

Consider a line **XY** to be hinged at the fixed point **O**, and capable of turning about it (Fig. 170). It cuts the fixed line **AB** in the point **P**. Let the line **XY**

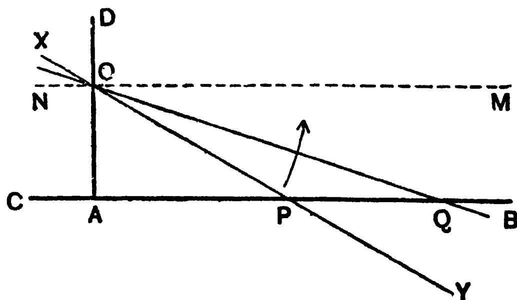


Fig. 170

be turned gradually round **O** in the sense as shown by the arrow. The point of intersection (**P**) recedes further and further from **A**; and when the line **XY** comes to the position, **NM**, as indicated by the dotted line,

the point of intersection goes to **an infinite distance from A or O and can never be reached.**

The angle **APO** between the *directions* **BA** and **YX** becomes smaller and smaller as the line **XY** turns more and more round **O** ; and in the limit when the point of intersection goes to infinity, **this angle vanishes,** and has its vertex at infinity.

Assuming for the present that an exterior angle of a triangle is equal to the sum of the interior opposite angles[1], we see (by joining **OA**) that,

$\angle OAC = \angle AOP + \angle OPA$, (from $\triangle$ AOP)

$= \angle AOQ + \angle OQA$, (from $\triangle$ AOQ)

and so on.

In the limit when **XY** comes to the position **NM**, $\angle AOP$ becomes $\angle AOM$, and $\angle OPA$ becomes zero, and we get,

$\angle OAC = \angle AOM$

$=$ Vert. opp. $\angle NOD$ (*Theor.* 3)

Thus the lines **MN** and **BC** make equal angles with, and on the same side of, the line **AD** ; hence **MN** and **BC are parallel** [§ 70].

Thus parallel straight lines may be regarded in either of the following ways :—

(i) They do not meet;

(ii) They meet, but the point of meeting is at infinity (i.e., however far you may travel along one of the lines, you do not reach this point).

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1. This truth may be *experimentally* verified in a number of cases by measurement or by placing the angles together as in § 46, Fig. 118 *Scientific proofs* (one of which will be independent of the theory of parallels) will be given later (see § 93, Note 1).

75. Horizontal Planes. Horizontal lines. If you go out of doors and take your stand on a wide *level* tract of land, the sky will appear to be a vast dome whose base seems to rest on the ground meeting it in a large circle. This circle is called the **Horizon,** and the surface of the level ground bounded by the Horizon gives you a rough idea of a particular sort of plane called a **level** or **horizontal plane.** The floor of a room, the top of a table, the top and the bottom of a rectangular box resting on the floor, the surface of water[1] in a basin or a pond, are examples of level or horizontal planes. All lines traced on a horizontal plane are **horizontal lines,** or simply **horizontals.** Cite a few more instances of horizontal planes.

Give examples of planes which are not horizontal.

What do you mean by an *inclined plane ?*

Take a sheet of cardboard. Hold it (i) in a horizontal position, (ii) in a non-horizontal position. How many different positions are possible in case (i), how many in case (ii) ?

Test of a level surface.

You can generally tell, by simply looking at a surface, whether it is level or not. This is a crude method, for, if there is a small amount of *slope,* you may not be able to direct it by simply looking at the surface. A better way will be to pour some water

1. The surface of the ground however smooth, and the surface of water at rest in a tank are not exactly planes. They are really slightly curved, being portions of very large spheres whose centre is the centre of the earth.

on the surface ; if the water begins to run down in any direction the surface is not horizontal or level. A more scientific way is to use what is called a spirit level[1],

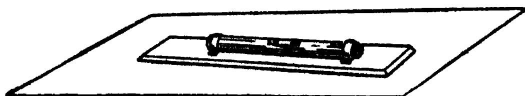


Fig. 171—A spirit level.

such as is used by a carpenter or a mason. Spirit levels are available in the market in various forms ; one such form is shown in Fig. 171.

76. **Lines perpendicular to a given plane.**

Hold a pencil upright on the table and draw a number of straight lines on the table through the foot of the pencil. Test the angles which the pencil makes with these lines, with a set-square. If the pencil has been held correctly you will find these angles to be all right angles.

The pencil is said to be perpendicular to the plane of the table. If a number of pencils were held *upright* on the table at different places, they would all be perpendicular to the plane of the table.

If a straight line meets a plane in a certain point O *and is perpendicular to all lines drawn in the*

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1. It consists of a vessel mounted length-wise on a base with a plane bottom (Fig. 171). The vessel contains spirit with a bubble of air floating on it. There is a glass covering at the top of the vessel, through which the position of the bubble may be observed. The instrument is placed on the surface to be tested, in various positions. If in each position the bubble lies in the middle as shown in (Fig. 171) the surface is horizontal. Students should familiarise themselves with the use of the instrument by actually testing various flat surfaces to see whether they are level.

plane through the point **O**, *it is said to be perpendicular to the plane.*[1]

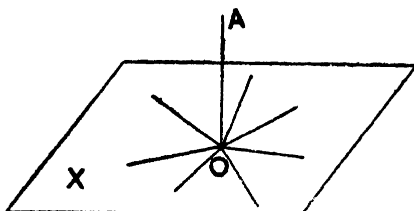


Fig. 172

Thus the line **OA** is perpendicular to the plane **X** (Fig. 172.)

Let us suppose, two pencils are held upright on the table, and a sheet of cardboard is placed just touching one of the pencils (say **A**) *along its whole length*

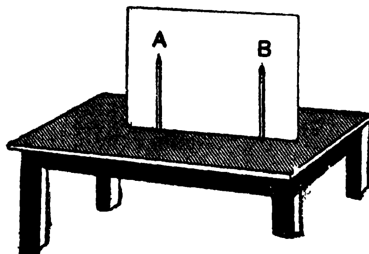


Fig. 173

(Fig. 173). Let the sheet of cardboard be then gently turned round this pencil (*without in any way disturbing its position*) and brought nearer and nearer to the second pencil, the *pencil A always remaining in contact with the sheet.*

1. It may be proved that if the line meeting the plane at **O** is perpendicular to each of *two* straight lines drawn on the plane through **O**, then it will be perpendicular to *all* straight lines drawn on the plane through **O**.

It will be found that the sheet will ultimately come to a position in which it will also touch the pencil B, *along its whole length, without disturbing its position.* This experiment shows that *two pencils held upright on the table lie in one plane* (the plane of the sheet of cardboard in its present position). We may generalise this result and say that *if two straight lines are both perpendicular to the same plane, they are coplanar, that is to say, they lie in one plane.*

Now mark two lines on the sheet of cardboard one along each pencil. Remove the sheet and examine these lines as in § 70, Fig. 160. You will find that any perpendicular to one of them is also perpendicular to the other, and that the perpendicular distance between the lines is the same at all points. This shows that *these lines do not meet.*

It will thus be seen *that if two straight lines are both perpendicular to the same plane, they are coplanar and do not meet ; in other words, they are parallel.*

Thus if a number of straight lines are all perpendicular to the same plane they are parallel to one another.

77. **Vertical lines and vertical planes.** Any line perpendicular to a level or horizontal surface is called **a vertical line** or simply a **vertical.** When an object (e.g. a telegraph post, a tower, a person etc.) stands *upright* on a level piece of ground, we say it is vertical. (Have you heard of the Leaning Tower of Pisa ? Why is it called leaning ?). If you take your stand at the foot of an upright telegraph post and look towards its top you will be looking *vertically up* or in the *vertically upward direction.* On the other hand, if you look down

towards your toes, you will be looking *vertically down* or in a *vertical downward direction.* All falling bodies (e.g. a piece of stone dropped from the top of a building) *fall in a vertically downward direction.* If a ball is suspended by a thread, *the thread takes up the vertical direction.*

Such a thread is called a **plumb line** (Fig. 174). A plumb line[1] indicates the vertical line through the point from which it is suspended.

Any plane passing through a vertical line is called a **vertical plane** (e.g. the walls of a room).

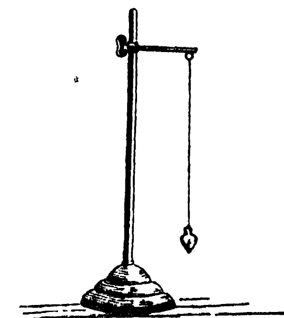


Fig. 174—Plumb line.

EXERCISE XXXI.

1 How many horizontal planes pass through (i) a given point, (ii) a given horizontal line ?

2 How many vertical planes pass through (i) a given vertical line, (ii) a given point ?

3. Point out (i) the vertical lines, (ii) the horizontal lines on a wall.

4. How many vertical lines can you trace on a wall ? How many horizontal lines ? Can you trace a line on the wall, which is neither vertical nor horizontal ?

5. Can you make a vertical plane pass through a given horizontal line ?

6. Can you make a vertical plane pass through a straight line which is neither vertical nor horizontal ?

1. "Plumb line, a line to which a mass of *lead* is attached to show the perpendicular"...... ..[L *Plumbum*, lead]—*Chambers's Twentieth Century Dictionary.*

7. How many horizontal lines pass through a given point? How many vertical lines?

8. If through a given point pass a vertical line and a horizontal line, what will be the angle between them?

9. 'A number of points are situated at the same level'. What do you mean by this statement?

10. What do you mean by the height of a point above the ground?

11. If a number of points are at the same height above the ground, will they be situated at the same level?

12. How would you test whether a certain post is vertical or not? [Use a plumb line.]

13 How would you test whether a certain wall is vertical or not? What test does a mason use?

14. How would you drive a stick into the ground so that it may be vertical?

78. Lines approximately parallel. If two points **A** and **B** are joined to a distant point **C**, such that **AC**,

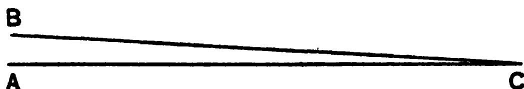


Fig. 175

BC are large compared to the distance **AB**, (Fig. 175) then the angle **ACB** will be small[1], and the lines **CA, CB** (including this small angle) will be *approximately* parallel (§ 74).

1. If a circle were traced with centre **O** with radius **OA**, it would meet **OB** in a certain point **N**, the arc **AN** of this circle would be a very small fraction of the circumference of the circle. Hence the angle which **AN** would subtend at the centre **O**, i.e., the ∠ **AOB** would be a very small fraction of four right angles, and therefore a small angle.

Thus if two persons were looking at the same star, they would be looking in practically the same direction, for a star is very far away

If two adjacent points **P** and **Q** of a circle (or a sphere) of a very large radius, were joined to the centre

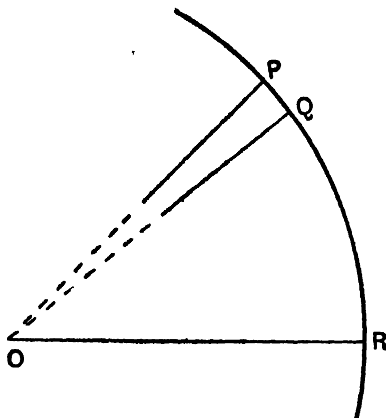


Fig. 176

O (Fig. 176) the angle **POQ** would be very small, and the radii **OP, OQ** might be regarded as approximately parallel. But if the points were not adjacent, but wide apart, such as **P** and **R**, the angle **POR** between the radii **OP, OR** would not be a small angle, and **OP, OR** would be far from being parallel.

79. **Are vertical lines, all over the Earth, parallel to one another ?** The Earth's surface is the outside of a very big sphere[1], a sphere whose radius is about 4000

1. If a circle is rotated about a diameter the circumference will trace out what is called a spherical surface. The centre and radius of the circle will also be the centre and radius of the sphere; all points on the surface of the sphere will be at the same distance from the centre, and this distance is the radius of the sphere [See Fig. 4 (c), page 1].

miles. Now it is a well-known property of a sphere, that if a straight line meets the surface of a sphere *perpendicularly*, it will pass through the centre of the sphere if produced, as shown in Fig. 177. Thus all verticals would pass through the centre of the Earth, if produced sufficiently. The verticals at the places **A**, **B**, **C**, are shown by the straight lines **AA′**, **BB′**, **CC′**, all standing upright (i. e. perpendicular to the sphere). In the vicinity of the place **A** a body would fall in the direction **A′A**, in the vicinity of the place **B** a body would fall in the direction **B′B**, and so on.

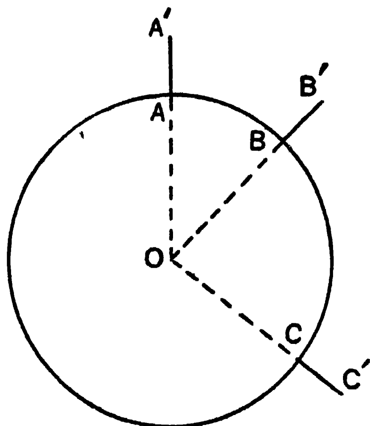


Fig. 177

These verticals are by no means parallel, not even *approximately* parallel. For example the vertical through a pole is *at right angles* to the vertical through a point

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The Earth is not *exactly* a sphere, it is more like an orange (flattened at the Poles, as you know) than a perfectly round ball.

The surface of the Earth is not smooth all over. In many places the ground rises into hills and high mountains, in other places it sinks to the beds of rivers or seas or oceans. But when you consider that the height of the highest mountain (Mt. Everest) is 29000 feet, and the depth of the deepest ocean (The Pacific Ocean) 36000 feet and the radius of the Earth 4000 miles, you will see that if the Earth were represented by a globe of radius of 1 foot, the height of Mt. Everest would be represented by one-sixtieth of an inch, and the depth of the Pacific Ocean by about a fiftieth of an inch. Considering therefore the large size of the Earth, the elevations of the mountains and the depressions of the seas would be insignificant.

on the equator, so that what is an upright direction for a man standing at one of the Poles is at right angles to what is an upright direction for a man standing at the Equator. If however, the distance between two places **P** and **Q** is small compared to the circumference (25000 miles) of the Earth (Fig. 178) the verticals through these places would include a small angle between them and would be virtually parallel. (§ 78). If, for instance the arc-distance between **P** and **Q** were 70 miles, the angle between the verticals **PP′**, **QQ′** i. e. the $\angle$**POQ** would be $\frac{70}{25000}$ of 360° i. e., about 1 degree. This being a small angle, the verticals **PP′** and **QQ′** would be *approximately* parallel. If **P** and **Q** were *nearer* to each other the verticals through **P** and **Q** would be *more approximately* parallel.

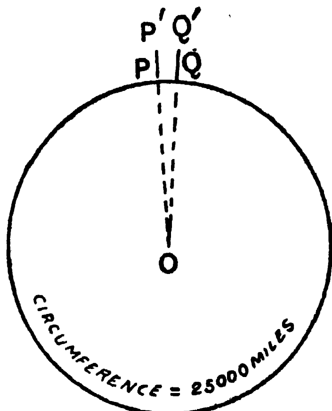


Fig. 178

DIRECTIONS.

80. The Principal directions :—North, South, East, West. What do you mean by such statements as the following :—

Harrison Road runs *east* and *west*. The *north* (or northern) wall of a room. The *Western* Block of the Senate House. The Office of the Phœnix Assurance Company Limited, is situated at 28, Dalhousie Square *West*. Two miles from here due *south* is the Hotel, and the Post Office is just a mile from the Hotel due *east*.

The terms North, South, East, West, occurring in the above statements indicate, as you are all aware, certain directions. What directions are these and how are they determined ? You may perhaps say that as the Sun rises in the east, we look eastward whenever we look at the rising Sun.

If you watch the rising Sun from your window from day to day, you will find that it does not rise at the same place all the days of the year. On a certain morning you may see the sun rising by the side of a certain mango tree, a few days later you will find it rising at a different place, that is to say, you will see it in a different direction. But the direction that we shall call East must be a *definite* direction, it must not be one direction one day, and another direction another day. Such a direction you cannot have from the rising Sun.

Can we appeal to the stars to give us our directions ? Note the position in the sky of a particular star, say at

about 9 o'clock on a certain fine night, and mark carefully its position relative to the surrounding stars, and also its brightness, so that you may recognise it afterwards. If you take another observation of that same star some two or three hours later, you will find that it has changed its position, it is not in its old position in the sky. Thus you do not see it in a constant direction. Such will be the case with other stars. But there is one star, called the Pole-star, which appears to be stationary[1] in the sky, and which may be recognised from its position relative to the other stars which all appear to move round this star. The star (the Pole-star) may give us our directions.

Go out of doors on a fine night with two or three friends and take your stand on a level piece of ground. Drive sticks or pegs into the ground to mark the places, **A, B, C** (Fig. 179) where you may be standing. Then begin to walk forward, all of you, with your eyes fixed at the Pole-star.[2] On going a few steps, drive pegs again at the places **D, E, F** where you may be standing now

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1. It is not exactly stationary, but the motion is so small, that it is *almost* stationary; the true point round which the stars appear to move, the celestial Pole, is almost coincident with this star and hence the name 'Pole-star.'

2. For this purpose it is necessary that you should know which one of the stars is the Pole-star. You may come to know it from a few observations of the stars under proper guidance by noting the position it occupies relative to some prominent neighbouring stars. On this point, as well as on the whole subject of directions you are advised to read '*An Introduction to Practical Geography*' by Simmons and Richardson (Macmillan & Co. Ltd.)

The straight lines **AD, BE, CF** indicate the paths along which you have walked. These lines will be found to be parallel[1]; this shows that *you have all walked in the same direction.*

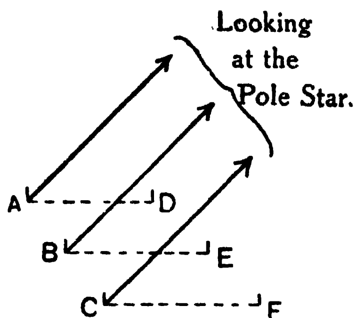


Fig 179

If you return to the same place at some other hour of the night and commence walking as before from the positions **A, B, C** respectively with your eyes towards the Pole-star, you will find that you have gone over the *same* paths **AD, BE, CF.** If you do the same thing on any other night and at any time of the night, you will be retracing the same paths, and walking in the same old direction.[2]

This definite direction which you may determine at any place on any fine night and at any time of the night by simply walking on level ground with your eyes directted to Pole-star, is called the **Northern** direction, and the opposite direction, the **Southern** direction. If you stand looking north-ward, the direction (at right

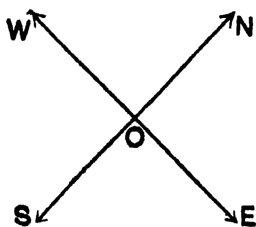


Fig. 180

1. When you are looking at the star you are all looking in the same direction, for the star is very far away (it may be millions of millions of miles) See § 78.

2. The reason for this is that the Pole-star is always at the same place in the sky.

angles to the northern) which lies to your right is the **Eastern** direction, and that which lies to your left is the **Western** direction. Evidently the East and the West are opposite directions (see Fig. 180).

81. **Other directions.** It should be observed that North, South, East, West are all horizontal directions.[1] There are many *other* horizontal directions, and all these may be definitely expressed with reference to the four standard directions. Thus the direction mid-way between the North and the East is the North-East (Fig. 181). It makes an angle of 45° with the North on the side of the East, or an angle of 45° with the East on the side of the North. In the same way you have, the South West (S. W.), North West (N. W), and South-East (S. E.) It is evident that N. W. and S. E. are opposite directions, so also N. E. and S. W.

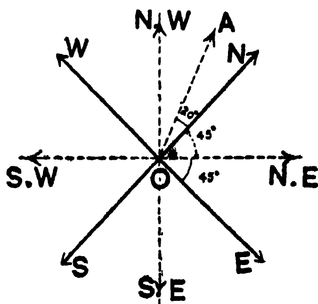


Fig. 181

Any horizontal direction **OA** may be expressed with reference to the standard directions (N, S, E, W) in the following way. Suppose the angle **AON** is 20°, we may say that the direction **OA** is 20° West of North. The same direction may also be expressed as 70° North of West.

1. The following story was narrated to the author by a friend of his. A boy was once asked to point out the southern direction. He pointed his finger towards his toes, indicating the downward direction. This mistaken notion no doubt arises from his use of maps, without having a clear idea of directions.

The direction which we call North-East (N. E.) is 45° North of East [45° N. of E.], or 45° East of North (45° E. of N.)

Ex. 1. The figure on the margin indicates the principal directions. Take any point and draw lines through it in the following directions :—**N.**, **S.**, **E.**, **W.**, **N. E.**, **S. E.**, 20° **N.** of **E.**, 35° **S.** of **W.**, 27° **W.** of **N.**

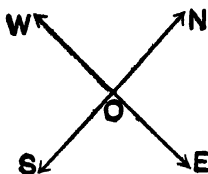


Fig. 182

Ex. 2. Name the directions opposite to the following :—**N. W.**, **S. W.**, 20° **N.** of **E**, 35° **S**, of **W.** 27° **W.** of **N.**

Ex. 3. Find the angle between the directions, (i) **S. W.** and **N. W.**, (ii) **N. E.** and **N. W.**, (iii) **N. E.** and **S. W.**, (iv) **S. W.** and 10° **N.** of **E.**, (v) 25° **E.** of **S.** and 10° **W**, of **N.**

Ex. 4. If **A** sees **B** 18° **E.** of **S.** in what direction will **B** see **A**?

82. *Other methods of determining the principal directions.* If you were to depend upon the Pole-star only for your North, you could not determine it by day time or on a foggy or cloudy night. It is known that the shadows of objects cast by the *mid-day* sun all extend North and South. This fact may be utilised in determining the North-South line. Another method is based on the use of Mariner's Magnetic Compass. For a detailed account of these methods the student is referred to '*An introduction to Practical Geography* by Simmons and Richardson.

83. **Directions other than horizontal.** If you stand at the foot of a telegraph-post and look towards its top you will be looking in a *vertically upward direction.*

When you observe a boat from the bridge, just below you, you will be looking in a *vertically downward direction.* When you stand on a bank of a river and observe the top of a tree on the opposite bank, you will be looking in a direction which is neither vertical nor horizontal. How will you express this direction ? Take a pair of compasses and point one of the legs to the top of the tree keeping the other leg *horizontal.*[1] The angle between the legs will give you what is called the **angular elevation** of the top of the tree above the level of your eyes. This angle is called the **angle of elevation,** and you are said to *observe the top of the tree at this angle of elevation.* What angle does the direction in which you see the top of the tree, make with the vertically upward direction ?

In the same way, when you observe a point lying below the level of your eyes you observe it at **an angle of depression.** This angle is found in the same manner as the angle of elevation by pointing one of the legs of the compasses towards the point keeping the other leg *horizontal.* Fig. 183 shows the elevation of the top of a telegraph-post as observed from a point **E** on the ground[2]; the angle **CEB** is the angle of *elevation.*

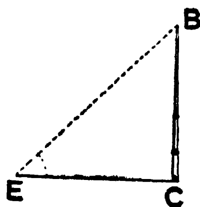


Fig. 183

1. The legs of the compasses should be in the vertical plane passing through the eye and the top of the tree.

2. The eye is supposed to be placed at the point **E** on the ground.

Fig. 184, shows the depression of the foot of a tree as observed from the top of a tower ; the angle QPA is the angle of *depression.*

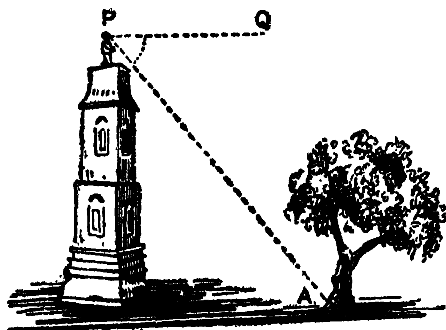


Fig. 184

It should be noted that EC and PQ are horizontal lines lying in the vertical planes through EB and PA respectively.

The angular elevation or depression of a point is usually measured with the aid of an instrument called the **Theodolite.** The angle between two horizontal directions may also be accurately determined with this instrument.

DIAGRAMS DRAWN TO SCALE.

84. We have already seen (§ 11) how large lengths or distances may be represented on paper by lines drawn on a reduced scale. We shall now see how the positions of various objects relative to one another, may be indicated by a **diagram drawn to scale.** Let us

take an example. *Suppose that in a certain town, the Post Office lies 2 miles north of the school, the Railway station 4 miles south-east of the Post Office and the Town Hall is due east of the school and due north of the Station.* How are we to draw a diagram on paper, showing the positions of these objects, relative to one another? In the first place we must choose a scale for representing the distances; in the second place we must choose an arrow to represent one of the principal directions, say the North: the other directions will then be determined. (see § 80, 81).

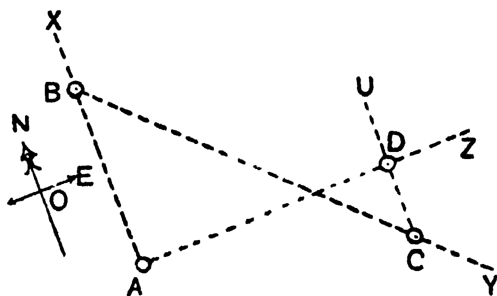


Fig. 185

Let us adopt a scale of $\frac{1}{2}$ inch to one mile, and choose the arrow **ON** to represent the north direction; then **OE** (perp. to **ON**, on the right) will represent the East, and the directions $\overline{\mathbf{NO}}$, $\overline{\mathbf{EO}}$,[1] will be the South and the West respectively (Fig. 185).

Let us now mark a point **A** on the paper, and *let it represent the position of the school* (Fig. 185). Through **A** draw a line **AX** parallel to **ON** to indicate the north

1. A bar (——) is often placed to indicate a direction.

direction from **A**, and cut off **AB** equal to 1 inch to represent a distance of 2 miles on the scale adopted. The point **B** *will represent the position of the Post Office.* From **B** draw a line **BY** such that the ∠**ABY** = 45°, **BY** will indicate the south-east direction from **B**; from **BY** cut off **BC** equal to 2 inches, to represent a distance of 4 miles. The point **C** *will represent the position of the Railway Station.* Lastly, from **A** draw **AZ** parallel to **OE**, and from **C** draw **CU** parallel to **ON**. The point **D** in which **AZ** and **CU** meet, will be east of **A** and north of **C**, and *will therefore represent the position of the Town Hall.* It should be noted that the data are just sufficient to fix the construction.

Once the diagram has been constructed we may derive much useful information from it. If we measure **AD**, **CD** with a ruler, we see that **AD** = 1·4 in., and **CD** = 0·4 in. On the scale adopted (viz., $\frac{1}{2}$ in. = 1 mile.), 1·4 in. represents a distance of 2·8 miles, and 0·4 in., a distance of 0·8 mile. *Thus we come to know that the Town Hall is about 2$\frac{4}{5}$ miles from the school, and about $\frac{4}{5}$ mile from the Station.* In the same way, by measuring **AC** we may obtain the distance of the Station from the school, and by measuring the ∠**DAC** we can know the direction in which the Station lies from the school.

85. We shall consider one more example to illustrate the use of diagrams drawn to scale.

How to determine the breadth of a river. Choose two positions **A** and **B** on the bank of the river (near the water-edge) from which you can observe a tree **T**, (or any other *stationary* object) on the opposite bank

in different directions (see Fig. 186).[1] Mark these positions by driving pegs into the ground, and measure the distance between the two positions by walking from one peg to the other, and counting the number of paces, or by stretching a string between the pegs and measuring its length afterwards, or in any other way. Now take your

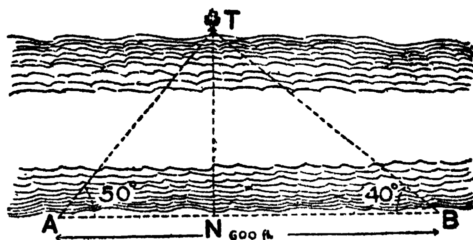


Fig. 186

stand at the position **A**, and find the angle which the direction **AT** (in which you observe the tree) makes with the direction **AB** i.e., ∠**BAT**. You may determine this angle[2] by holding a pair of compasses horizontally, and pointing one of the legs towards the tree and the other towards **B**. Then take your stand at the other position **B**, and determine in a similar way, the angle which the direction **BT** makes with the direction **BA** i.e., ∠**ABT**. Suppose you find **AB** = 600 ft., ∠**TAB** = 50°, ∠**TBA** = 40°. Now take a scale of $\frac{1}{6}$ inch to 100 ft., and draw a line **PQ** = 1 inch to represent a distance of 600 ft. (Fig. 187.)

1. The river is shown on a reduced scale.

2. The angle can be measured more accurately with the aid of an instrument called the Theodolite.

Through **P** draw **PX**, making $\angle$**QPX** = 50°, and through **Q** draw **QY** making $\angle$**PQY** = 40°. Let **PX** and **QY** meet in **O**. From **O** draw **OR** perpendicular to **PQ**. Then **OR** will represent the breadth, **TN**, of the river on a scale of $\frac{1}{6}$ inch to 100 feet.

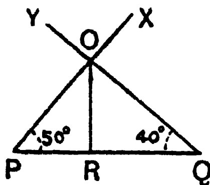


Fig. 187

If you measure **OR** you will find it to be $\frac{1}{2}$ inch. Hence the breadth of the river is $(\frac{1}{2} \div \frac{1}{6} \times 100)$ feet i. e. 300 feet.

86. The illustrations given in the two preceding articles show the utility and importance of diagrams drawn to scale. The map of a town or a district, the plan of a building or a play-ground are examples of diagrams drawn to scale[1]. A glance at a map or a plan gives you an idea of the relative positions of the various objects situated in the area, which the map represents. Making a copy of the map on tracing paper and investigating it with a ruler, a pair of set-squares, and a protractor you may gather much useful and interesting information.

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1. You will see from the illustrations given in §§ 84, 85, how a limited number of data (obtained before-hand by direct measurements) are sufficient to fix the diagram. Once the diagram has been constructed from the data, you may derive from it distances or angles which were not, or could not be, obtained before by direct measurement.

When we infer results regarding the original object, from the map or diagram representing it, we make two assumptions, viz., (i) all the lengths and distances on the map are proportional to the corresponding lengths or distances in the original object, (ii) and the angles on the map are equal to the corresponding angles in the original object. These assumptions, originally derived

Here is a plan of a part of Calcutta (Fig. 188). Copy it on a piece of tracing paper and answer the following questions :

1. In what direction does Harrison Road run from Sealdah Station to the College Street crossing ?

2. In what directions do the following streets or roads run ?

(i) Dharamtola Street, (ii) Bowbazar Street, (iii) Circular Road, (iv) College Street, (v) Chittaranjan Avenue.

3. Find the distances (in a straight line) of the following places from the Presidency College. (i) Howrah Station, (ii) Sealdah Station, (iii) Government House, (iv) Victoria Memorial Hall, (v) Eden Garden.

4. What is the distance and bearing of the Victoria Memorial Hall from Chandpal Ghat ?

5. What is the length of Harrison Road ?

6. What is the distance from the Presidency College to the Howrah Bridge along Harrison Road ?

7. What is the span of the Howrah Bridge ?

8. What is the breadth of the river Hoogly a mile to the south of the Howrah Bridge.

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from intuition, rest upon a fundamental geometrical truth : "If two triangles have the angles of the one equal to those of the other, each to each, then the sides of the one are proportional to the corresponding sides of the other. On this subject of *'drawing to scale'* the student may read with profit D. B. Mair's *'A school course of Mathematics'* (Clarendon Press, Oxford), Chap. IX.

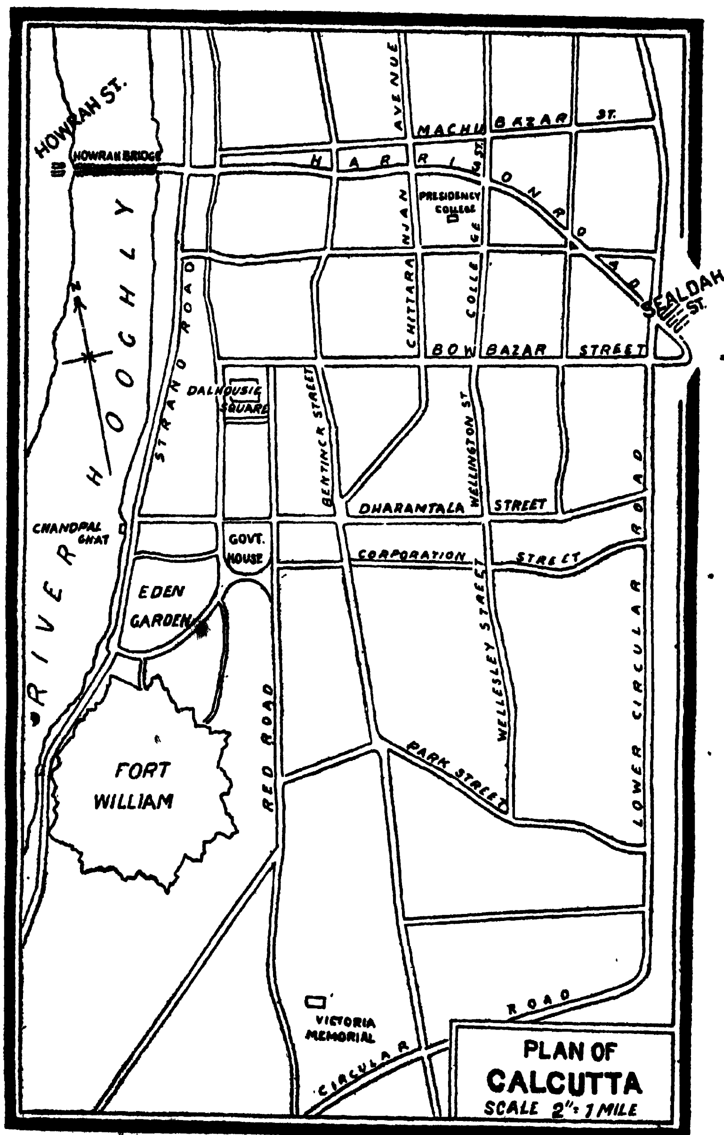

HOWRAH ST.
HOWRAH BRIDGE
RIVER HOOGHLY
N
MACHUA BAZAR ST.
HARRISON ROAD
AVENUE
CHITTARANJAN
PRESIDENCY COLLEGE
COLLEGE ST.
SEALDAH ST.
BOW BAZAR STREET
STRAND ROAD
DALHOUSIE SQUARE
BENTINCK STREET
WELLINGTON ST
DHARANTALA STREET
CHANDPAL GHAT
GOVT. HOUSE
CORPORATION STREET
EDEN GARDEN
WELLESLEY STREET
LOWER CIRCULAR ROAD
FORT WILLIAM
RED ROAD
PARK STREET
VICTORIA MEMORIAL
CIRCULAR ROAD
PLAN OF
CALCUTTA
SCALE 2" = 1 MILE

Fig. 188

EXERCISE XXXII.

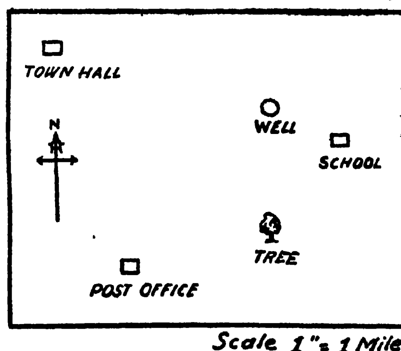
N. B.—In drawing diagrams, the student must *choose a scale* for representing distances, and an *arrow* to indicate the North direction, whenever these are not assigned in the problem.

1. A man walks 4 miles due East and then 3 miles due North. Map his path on a scale of $\frac{1}{2}$ inch to one mile.

2 A man starts from a place **A** and walks 5 miles east, then 3 miles north, then 2 miles west, then 1 mile south, finally reaching a place **B**. Draw a plan of the route taking 1 cm. to stand for a mile. Find the distance between the two places, and the bearing[1] of **B** from **A**.

3 Make a map on the scale of $\frac{1}{2}$ inch to a mile, showing the positions of the places **P**, **Q**, **R**, **S**, given that **Q** is 4 miles north of **P**, **R** is 3 miles west of **Q**; and **S** is 5 miles south of **R**. What is the distance from **P** to **R** on a straight road ?

4. **A** and **B** are two places distant 5 miles and 4 miles respectively from the place **C**. As viewed from **C**, the places **A** and **B** are in the directions 30° **E.** of **N.** and 40° **S.** of **E.** respectively. Map the positions of the places on a suitable scale and find the distance and the bearing of **B** from **A**.



Scale 1" = 1 Mile

Fig. 189

5. From the above map (Fig. 189) find roughly the bearing

1. The bearing of **B** from **A** means the direction in which **B** lies with respect to **A**.

and distance of (i) the Post Office from the school (ii) the Town Hall from the Post Office (iii) the well from the Post Office, from the school (iv) the tree from the well.

6. Two persons **A** and **B** start from the same place. **A** walks 3 miles North, and then 5 miles North-West; **B** walks 5 miles North-West and then 3 miles North. Do they come to the same place?

7. An ice-bound ship drifts as follows; 20 miles North, 5 mls. **N. E.**, 15 mls. **S. E.**, 35 mls. **S.**, 25 mls **N.W.**, 10 mls. **W.**, 12 mls, **S. W.**, and 30 mls. **E.** Map its course on a scale of an inch to 10 miles.

8. I can see two houses, one due **S.**, $\frac{3}{4}$ mile away, the other due **N. E.** $\frac{1}{2}$ mile away. How far apart are the two houses?

9. A man sees a tree 15° **N.** of **E**, and after walking 300 yards 35° **S.** of **E**, he sees it due **N.** How far is he from the tree in his second position?

10. Two persons start from the same place at 6 A. M.; one walks due **N.** at the rate of 3 miles per hour, the other walks **S. E** at the rate of $2\frac{1}{2}$ miles per hour. What will be the distance between them at 7-30 A. M.?

11. Two steamers **P** and **Q** are 500 yards apart, **Q** lying south of **P.** A station on the shore is south-west of **P** and west of **Q**. How far is each of the ship from the station?

12. Draw a plan of a rectangular field 400 yards long and 200 yards broad on a scale of $\frac{1}{2}$ inch to 100 yards. Find the distance between two opposite corners.

13 Two places **A** and **B** are 600 yards apart, **B** lying north of **A.** A man passes close to **A** due East, and after 6 minutes finds **B** lying north-west of him. Find the rate at which the man is walking.

14. A man standing on one bank of a river sees a tree on the other bank in a direction 30° **E.** of **N.** He walks 800 feet along the bank and then sees the tree just opposite to him. If the river flows east and west find the breadth of the river.

15. A man walking along a straight road observes a house in a direction 35° with the road. Walking a mile further he sees the house in a direction of 72° with the road. Find the distance of the house from the road.

16. The angular elevation of the top of a tower from a point on the ground, 200 feet from the base of the tower, is 40°. Find height of the tower.

17. A telegraph post 30 feet high stands on the ground; find the angles of elevation of its top, and its middle point as seen from a point on the ground, 100 feet from the foot of the post.

18. A man 6 feet high, standing at a distance of 25 feet from a tree sees its top at an elevation of 30°. Find the height of the tree.

19 A man walks 300 feet from the base of a chimney and sees its top at an elevation of 20°. Find the height of the chimney.

20. If a vertical stick 5 feet long casts a shadow 8 feet in length, find the elevation of the Sun.

21. A man on a river-bridge sees a boat at a depression of 25°. If the height of the bridge is 45 feet, and the boat is travelling at the rate of 2 miles an hour, how long will it take the boat to reach the bridge?

22. A man sitting on a cliff 300 feet above sea-level sees a boat at a depression of 32°. What is the distance of the boat from the man?

23. Two lamp-posts, **P** and **Q** of the same height are situated 200 feet apart. A man standing between them, at a distance of 50 feet from **P**, observes the top of **Q** at an elevation of 10°. Find the height of the posts, and the angle of elevation at which the man sees the top of **P** from the same position.

24. A vertical pole is stuck into the ground at the centre of a circular field; the top of the pole is 60 feet above the ground and is observed at an elevation of 10° from a point on the circumference. Find the radius of the field.

25. A vertical pole is situated at the top of a tower 56 feet high. A man sees the top of the tower at an elevation of 30°, and the top of the pole at an elevation of 40°. If the height[1] of the man be 6 feet, find the length of the pole and the distance of the man from the foot of the tower

26. From the top of a house 60 feet high, the elevation of the top of a tower is observed to be 40°. From the ground floor the elevation is 56°. Find the height of the tower.

27. From the top of a cliff the depressions of two buoys in the same vertical plane are 40° and 18°. If the buoys are 900 yards apart, find the height of the cliff.

28. A man on a river bridge sees a boat (sailing towards the bridge at the rate 2 miles per hour) at a depression of 10°; after 1 minute 20 seconds the boat is observed at a depression of 26°. Find the height of the bridge.

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1. If the height of the observer is not particularly stated, the eyes of the observer must be supposed to be on the ground.

CHAPTER VI.

THEOREMS ON PARALLELS AND ALLIED THEOREMS AND PROBLEMS.

87. We have already defined two straight lines to be **parallel** *if they lie in the same plane but do not meet however far they are produced either way.* We have also studied some of the properties of parallel lines in an experimental way. The student is advised to re-read Chapter V, §§ 70-74.

We now proceed to treat the properties of parallels in a more scientific way. For this purpose it is necessary for us to know the meanings of certain terms which will be used in this connection.

When a straight tine **EF** falls across two other straight lines **AB,CD**, as in Fig. 190, eight angles are formed. These are numbered 1,2,⋯,8 in the figure.

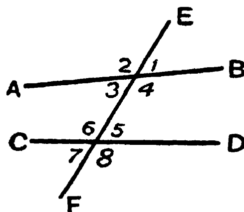


Fig. 190

The angles, 1 and 5, are called **corresponding angles.** So also the angles 2 and 6 are another pair of corresponding angles. There are two other pairs of corresponding angles, viz., 4 and 8, and 3 and 7.

The angles 3 and 5 are called **alternate angles** ; 4 and 6 form another pair of alternate angles.

The angles 1, 2, 7, 8 are also called **exterior angles** and the angles 3, 4, 5, 6, **interior angles**; of the two angles 4 and 8, 8 is often referred to as the **exterior** angle, and 4 as the **interior opposite** angle on the same side of **EF**. Evidently an exterior angle and the interior opposite angle form a pair of corresponding angles.

A straight line falling across two or more straight lines is called a **transversal** Thus the straight line **EF**, falling across the two straight lines **AB**, **CD** is a transversal (Fig. 190).

THEOREM 13 A. [Euclid. 1, 27].

88. If a straight line cuts two other straight lines so as to make a pair of alternate angles equal, the straight lines are parallel.

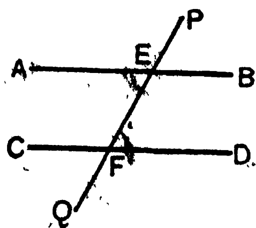


Fig. 191

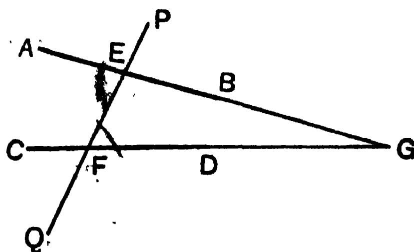


Fig. 192

The straight line PQ cuts the straight lines AB, CD so as to make the alternate angles AEF, EFD equal (Fig. 191); *to prove that* AB, CD are parallel.

Proof. If **AB, CD** are not parallel, they will meet if produced either towards **B** and **D**, or towards **A** and **C**. If possible, let them meet when produced towards **B** and **D**, at the point **G**, as in Fig. 192.

Then the ∠**AEF** is an exterior angle of the △ **EFG**,

∴ ∠**AEF** is greater than the interior opposite ∠**EFD**. (Theor. 8).

But by hypothesis the ∠ˢ **AEF, EFD** are equal.

Hence **AB, CD** *cannot meet* when produced towards **B** and **D**, and form a triangle **EGF**, as in Fig. 192.

In a similar way it may be shown that **AB** and **CD** cannot meet when produced towards **A** and **C**.

∴ **AB and CD are parallel.**

THEOREM 13 B. [Euclid 1, 28.]

89. If a straight line cuts two other straight lines so as to make

(i) a pair of corresponding angles equal, or

(ii) a pair of interior angles on the same side of the cutting line, together equal to two right angles,

then the straight lines are parallel.

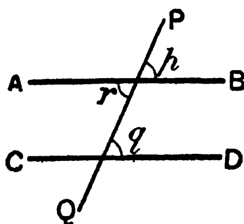


Fig. 193

(i) **PQ** cuts **AB** and **CD** so as to make the *corresponding* angles *p* and *q* equal (Fig. 193).

To prove that **AB** and **CD** are parallel.

Proof. $\angle p = \angle q$, (*Given*)

also $\angle p =$ vert. opp. $\angle r$,

$\therefore$ $\angle r = \angle q$;

and these are alternate angles;

$\therefore$ **AB and CD are parallel.** (*Theor.* 13A)

(ii) **PQ** cuts **AB** and **CD** so as to make the interi

angles q and s (on the same side of **PQ**) together equal to two right angles (Fig. 194).

To prove that **AB** and **CD** are parallel.

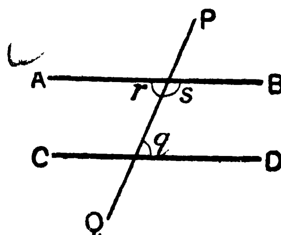


Fig. 194

Proof. $\angle q + \angle s = 2$ rt. $\angle^s$; (*Given*)

also $\angle r + \angle s = 2$ rt. $\angle s$; (*Theor.* 1)

$\therefore \quad \angle q + \angle s = \angle r + \angle s.$

Taking away $\angle s$ from both sides, we have $\angle q = \angle r$; and these are alternate angles,

$\therefore$ **AB** and **CD** are parallel.

Corollary. If two straight lines are each perpendicular to a third straight line, they are parallel.

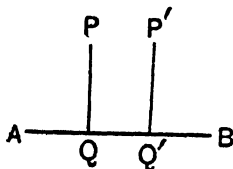


Fig. 195

PQ, P′Q′ are each perpendicular to **AB**. The interior angles **PQB** and **P′Q′A** being right angles, their sum is two right angles.

Hence **PQ, P′Q′** are parallel.

Ex. 1. Two straight lines **AC, BD** bisect each other at **O**. Show that **AB** is parallel to **CD** and **AD** is parallel to **BC**.

Ex. 2. **ABCD** is a quadrilateral. If **AB = CD**, and **AD = BC**, show that **AB** is parallel to **CD** and **AD** is parallel to **BC**.

PROBLEM 6.

90. Through a given point to draw a straight line parallel to a given straight line.

O is a given point and **AB** a given straight line.

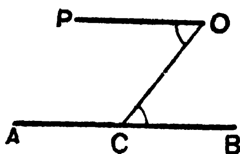


Fig. 196

Through **O** to draw a straight line parallel to **AB**.

Construction. Take any point **C** in **AB** and join **OC**. Using the construction of Problem 5 (§ 64), at the point **O** in the line **OC** make $\angle$**COP** equal to $\angle$**OCB**, and alternate to it. Then **OP** is parallel to **AB**.

The proof is left to the student.

Note.—This construction is based on the use of ruler and compasses only (See § 50). In actual practice parallels are drawn with the aid of set-squares as explained in § 71. The student is advised to repeat Exercise **XXX**, Exs. 1, 15, 16.

THEOREM 14. [Euclid 1, 29].

91. If a straight line cuts two parallel straight lines,

(i) alternate angles are equal;

(ii) corresponding angles are equal;

(iii) the interior angles on the same side of the cutting line are together equal to two right angles.

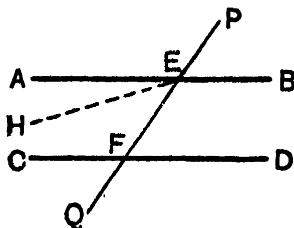


Fig. 197

AB and CD are parallel and PQ cuts them in E and F.

To prove that :—

(i) alternate ∠s AEF, EFD are equal,

(ii) corresponding ∠s PEB, EFD are equal,

(iii) the interior ∠s BEF, EFD are together equal to two right angles.

Proof. (i) If ∠AEF is not equal to ∠EFD, suppose ∠HEF is equal to ∠EFD.

As the alternate ∠s HEF, EFD are equal,

∴ EH is parallel to CD. (*Theor.* 13A)

but AB is also parallel to CD; (*Given*)

thus two intersecting straight lines **AB, EH** *are both parallel to* **CD**, which is impossible.[1]

∴ ∠**AEF** cannot be unequal to ∠**EFD.**

i.e., **∠AEF is equal to ∠EFD.**

(ii) It has just been proved that ∠**AEF**=∠**EFD**;

but ∠**AEF**=vert. opp. ∠**PEB**,

∴ ∠**PEB**=∠**EFD**,

i.e., *the corresponding* ∠s **PEB, EFD** *are equal.*

(iii) As **EF** stands on **AB**,

∴ ∠**AEF**+∠**BEF**=**2** right angles.

but by (i), ∠**AEF**=∠**EFD**;

∴ ∠**EFD**+∠**BEF**=2 right angles.

i.e., **the sum of the interior ∠s EFD, BEF is 2 right angles.**

Note. Theorem 14 is the converse of Theorems 13A, 13B.

---

1. It appears obvious that *two intersecting straight lines cannot both be parallel to a third straight line.* In reality, however, it is an assumption which we accept without proof, and is known as **Playfair's Axiom.** It may also be stated in a different form, '*Through a given point there can pass only one straight line parallel to a given straight line.*

THEOREM 15. [Euclid 1, 30.]

92. **Straight lines which are parallel to the same straight line are parallel to one another.**

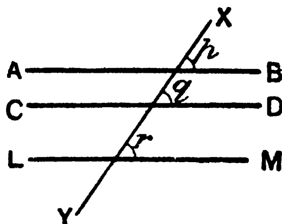


Fig 198

AB and **CD** are each parallel to **LM**.

To prove that **AB** is parallel to **CD**.

Draw any straight line **XY** cutting **AB**, **CD**, **LM** and forming with them corresponding angles p, q, r respectively.

Proof. Because **AB** is parallel to **LM**,

$\therefore$ $\angle p =$ corresponding $\angle r$. (*Theor*. 14)

Again because **CD** is parallel to **LM**,

$\therefore$ $\angle q =$ corresponding $\angle r$,

$\therefore$ $\angle p = \angle q$;

and these are corresponding angles,

$\therefore$ **AB** is parallel to **CD**. (*Theor*. 13B)

EXERCISE XXXIII.

1. Prove case (ii) of Theorem 14, *directly* without assuming case (i).

2. Prove case (iii) of Theorem 14, directly without assuming case (i) or case (ii).

3. If the ∠**PEB** in Fig. 197 be 60°, find all the angles in the figure.

4. **AB**, **CD** and **PQ**, **RS** are two pairs of parallel lines as shown in Fig. 199, and the angles are numbered 1, 2..., 16. Prove that the following pairs of angles are equal,—(1, 9); (4, 8); (7, 16); (1, 11); (6, 16); (5, 15); (2, 12); (8, 9); (3, 14); (5, 11); (6, 12); (7, 12); (2, 14) Point out all the angles in the figure which are equal to 2. If the angle 1 = 75°, find the values of all the other angles.

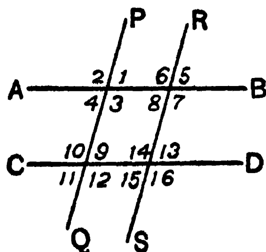


Fig. 199

5 *The opposite angles of a parallelogram are equal.*

6. *If one angle of a parallelogram is a right angle, then all its angles must be right angles.*

7. *Prove that the four angles of a parallelogram are together equal to 4 right angles.*

8. *If an angle has its arms respectively parallel to the arms of another angle, then the two angles must be either equal or supplementary.*

9. *If a number of lines are parallel, then any straight line drawn perpendicular to one of them, will be perpendicular to all of them.*

10. If a line, drawn parallel to the base **BC** of an isosceles triangle **ABC**, meets the equal sides **AB**, **AC** in **P** and **Q**, then the triangle **APQ** is isosceles.

11. From any point **P** (Fig. 200) in the bisector of an angle **BAC**, a line is drawn parallel to **AC**, to meet **AB** in **Q**. Show that the triangle **APQ** is isosceles.

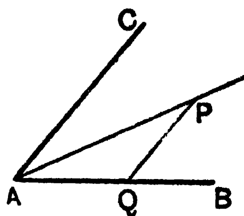


Fig. 200

12. From any point in the base **BC** of an isosceles triangle **ABC**, a perpendicular is drawn to the base, meeting the sides **AB**, **AC**, (produced if necessary) in **P** and **Q** respectively. Show that the triangle **APQ** is isosceles.

13. If the external bisector of an angle of a triangle be parallel to the opposite side, show that the triangle is isosceles.

THEOREM 16. [Euclid 1, 32.]

93. The three angles of a triangle are together equal to two right angles.

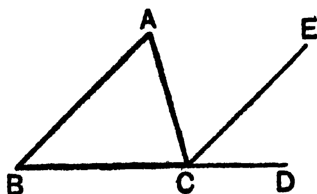


Fig. 201

ABC is a triangle.

To prove that the three angles **CAB, ABC, BCA** are together equal to two right angles.

Produce **BC** to **D** and through **C** draw **CE** parallel to **AB**.

Proof. Because **BD** meets the parallels **AB, CE,**

∴ ∠**ABC** = corresponding ∠**ECD**. (*Theor.* 14)

Because **AC** meets the parallels **AB, CE,**

∴ ∠**CAB** = alternate ∠**ACE**. (*Theor.* 14)

∴ ∠**ABC** + ∠**CAB** = ∠**ECD** + ∠**ACE** = ∠**ACD**.

Add ∠**ACB** to each side,

∴ ∠**ABC** + ∠**CAB** + ∠**ACB** = ∠**ACD** + ∠**ACB**.

= 2 right angles ;

That is to say, the three angles of the △ ABC are together equal to 2 right angles.

(Corollary. If one side of a triangle be produced the exterior angle so formed is equal to the sum of the interior opposite angles.)

This result has been directly obtained in course of the above proof.

It follows that an exterior angle of a triangle is greater than either of the interior opposite angles.

Note 1. Another proof of Theorem 16. Let **ABC** be a triangle. Suppose you start from the point **X** in the side **CA** (Fig. 202) and walk round the triangle in the sense indicated by the arrows, until you come back to **X**. At **A** you turn through an angle **P**, at **B** through an angle **Q**, and at **C** through an angle **R**. As you are now walking in the same direction as that in which you started you have turned through a total angle of four right angles.

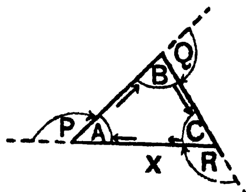


Fig. 202

Hence $\angle \mathbf{P} + \angle \mathbf{Q} + \angle \mathbf{R} = 4$ right angles,

Now $\angle \mathbf{A} + \angle \mathbf{P} = 2$ right angles,

$\angle \mathbf{B} + \angle \mathbf{Q} = 2$ right angles,

$\angle \mathbf{C} + \angle \mathbf{R} = 2$ right angles,

$\therefore \angle \mathbf{A} + \angle \mathbf{B} + \angle \mathbf{C} + \angle \mathbf{P} + \angle \mathbf{Q} + \angle \mathbf{R} = 6$ right angles.

But $\angle \mathbf{P} + \angle \mathbf{Q} + \angle \mathbf{R} = 4$ right angles.

$\therefore \angle \mathbf{A} + \angle \mathbf{B} + \angle \mathbf{C} = 6$ right angles $-$ 4 right angles

$= 2$ right angles.

Thus the theorem is proved, and it may be noted that this proof is independent of the theory of parallels.[1]

We may deduce that an exterior angle of a triangle is equal to the sum of the two interior opposite angles.

1. This proof is indicated in D. Mair's *School Course of Mathematics.*' Clarendon Press, Oxford.

For we have

$$\angle A + \angle B + \angle C = 2 \text{ right angles,}$$
$$\text{and } \angle A + \angle P = 2 \text{ right angles,}$$
$$\therefore \quad \angle A + \angle P = \angle A + \angle B + \angle C.$$
$$\therefore \quad \angle P = \angle B + \angle C.$$

Note 2. Theorem 16 may be proved experimentally in a very interesting way.

Take any paper-triangle **ABC** (Fig. 203) and fold the side **BC** upon itself in such a way that the crease **AD** passes through **A**. Open out the paper, and fold the corners **A**, **B**, **C** all on to **D**. Then the figure takes up a rectangular form **EFGH**. It will be

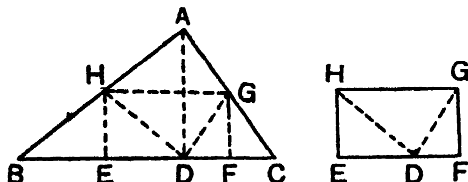


Fig 203

found that the $\angle$s **GDH**, **EDH**, **GDF** are the new positions of the $\angle$s **A**, **B**, **C**. But the $\angle$s **GDH**, **EDH** and **GDF** fill up two right angles. Hence the $\angle$s **A**, **B**, **C**, are together equal to two right angles.

Note 3. The following inferences from Theorem 16 should be noticed.

(i) *If two triangles have two angles of the one respectively equal to two angles of the other, then the third angle of the one will be equal to the third angle of the other.*

(ii) *The sum of any two angles of a triangle is the supplement of the third.* Thus if two angles of a triangle are given the third may be found at once, by subtracting the sum of the two given angles from 180 degrees.

(iii) **In a right-angled triangle, the two acute angles are complementary.**

(iv) If one angle of a triangle is equal to the sum of the other two, then the triangle is right-angled.

(v) The four angles of a quadrilateral are together equal to four right angles. The quadrilateral is divided into two triangles by a diagonal (Fig. 204). The four angles of the quadrilateral are together equal to all the angles of the two triangles and are therefore equal to four right angles.

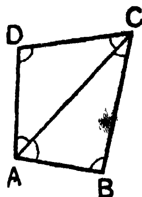


Fig. 204

EXERCISE XXXIV.

1. *The angles of an equilateral triangle are each twothirds of a right angle,* i.e., *60°.*

2. Show that in a right-angled isosceles triangle each of the acute angles is 45°.

3. In a triangle **ABC**,

   (i) $\angle A = 50°$, $\angle B = 70°$ find $\angle C$.

   (ii) $\angle A = 80°$, $\angle B = 40°$, find $\angle C$.

   (iii) $\angle A = \angle C = 70°$, find $\angle B$.

   (iv) $\angle A + \angle B = 102°$, and $\angle A = 2\angle B$, find all the angles **A, B, C**.

4. Each of the base angles of an isosceles triangle is double the vertical angle. Find the magnitudes of the angles.

5. One angle of a triangle is 60°; of the other two angles, one is 4 times as large as the other, find the angles.

6. If **A, B, C** are the angles of a triangle, show that $\frac{A}{2}$ is the complement of $\frac{B+C}{2}$.

7. In a quadrilateral **ABCD**, $\angle A = 80°$, $\angle B = 60°$, $\angle C = 110°$, find $\angle D$.

8. In a quadrilateral **ABCD**, $\angle A = 80°$, $\angle B = 60°$, and $\angle C = \angle D$, find $\angle C$.

9. If two angles of a quadrilateral are supplementary, the remaining two angles will also be supplementary.

10. If the exterior angles formed by producing a side of a triangle both ways be 120° and 80°, find all the angles of the triangle.

11. Find the sum of the exterior angles formed by producing the sides of a triangle *in order* (Fig 205)

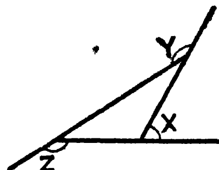


Fig. 205

12 Prove Theorem 16 by simply considering a line to be drawn through **C**, parallel to **AB** (Fig. 206), *without producing* **BC** *to* **D** as in Fig. 201.

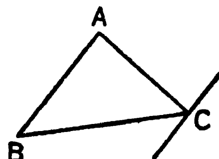


Fig. 206

13. If two straight lines are perpendicular to two given straight lines respectively, then the acute angle between the perpendiculars is equal to the acute angle between the given lines.

Two cases are shown in Fig. 207. (a), (b). **PQ, PR** are ⊥ to **AB, AC** respectively. Consider the right-angled △s **AQS, PRS.** In Fig. (a), ∠s **A, RPS** are each a complement

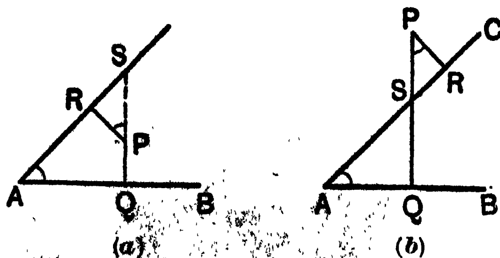


Fig. 207

of ∠**ASQ** (Note 3, (iv) § 93], hence they are equal. In Fig. (b),

∠A = complement of ∠ASQ and ∠P = complement of ∠PSR; but ∠s ASQ, PSR are equal being vertically opposite; hence ∠s A, P are equal.

14. If a straight line cuts two parallel straight lines and the two interior angles on the same side of the cutting line, be bisected show that the bisectors are at right angles.

15. If one side of a triangle be produced both ways, then the sum of the exterior angles so formed, minus that angle of the triangle, which is opposite to this side, is equal to two right angles.

16. ABC is a triangle; D is the mid-point of BC, and A is joined to D. Show that if AD = BD = CD the triangle is right-angled.

17. The extremities A and B of a diameter of a circle are joined to any point P on the circle. Show that the ∠APB is a right angle.

18. **ABC** is a right-angled triangle, ∠**A** being a right angle. **AD** is drawn perpendicular to **BC**. Show that ∠**B** = ∠**DAC**, ∠**C** = ∠**BAD**.

94. Two important corollaries to theorem 16.

Corollary I. All the angles of any rectilinear figure, with four right angles added to them, make up twice as many right angles as the figure has sides.

Let us take a rectilinear figure (**ABCDEF**), say, of 6 sides.

To prove that all the angles of this figure + 4 right angles = twice as many right angles as the figure has sides, i.e., 12 right angles in the present case.

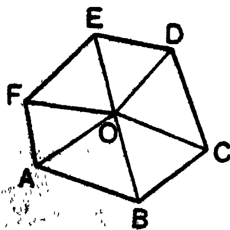


Fig. 208

Take any point O within the figure, and join it to the vertices. Then the figure is divided into as many triangles as the figure has sides (6 triangles in the present case).

The three angles of each triangle together = 2 right angles.

Hence all the angles of all the six triangles together
$= (2 \times 6)$ i.e. 12 right angles

But all the angles of all the triangles make up all the angles of the given figure together with the angles round the point O, which equal 4 right angles.

Hence all the $\angle^s$ of the figure + 4 right angles = 12 right angles, i.e., twice as many right angles as the figure has sides.

From the reasoning employed above it is evident that if a figure has n sides,

all the $\angle^s$ of the figure + 4 right angles = $2n$ right angles.

Note. If a polygon is *equiangular*, that is, has all its angles equal, we can at once calculate the value of each of its angles.

Suppose the polygon has n sides, and D is the value of each angle, then

nD + 4 right angles = $2n$ right angles.

nD = $(2n - 4)$ right angles.

$\therefore$ $D = \frac{2n-4}{n}$ right angles.

We know that a *regular polygon* is one which has all its sides equal and all its angles equal (§ 40). Each of the angles of a regular polygon of n sides is therefore, $\frac{2n-4}{n}$ right angles or $\left(\frac{2n-4}{n}\right) \times 90$ degrees.

Corollary II. If the sides of a convex rectilinear figure are produced in order, the sum of the exterior angles so formed is equal to four right angles.

At each vertex,

the interior angle + the exterior angle = 2 right angles.

Hence the sum of the interior angles + the sum of the exterior angles = twice as many right angles as the figure has *vertices*, i. e., twice as many right angles as the figure has sides (for a rectilinear figure has as many vertices as sides).

Fig. 209

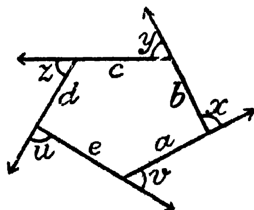
But by Cor. I,

the sum of the interior angles of the polygon + 4 right angles = twice as many right angles as the figure has sides.

Hence the sum of the exterior angles = 4 right angles.

A direct proof of Cor. II. Let the sides *a*, *b*, *c*, *d*... of the figure be produced *in order* (Fig. 210).

Take any point **O** and draw lines **OA, OB, OC, OD, OE** from **O**, parallel to the sides



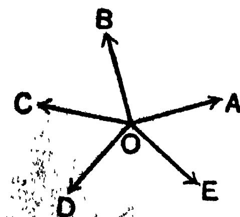


Fig. 210

to *b*, *c*, *d*, *e* of the figure, *in the directions in which*

the sides are produced. Because directions **OA, OB** are the same as the directions in which the sides a and b are produced,

$\therefore$ the ext. $\angle x = \angle AOB$;

similarly, $\angle y = \angle BOC$,

$\angle z = \angle COD$,

$\angle u = \angle DOE$,

and $\angle v = \angle EOA$.

$\therefore \angle x + \angle y + \angle z + \angle u + \angle v =$ sum of the angles round **O**.

$= 4$ right angles.

Note. Cor. I. may now be deduced from Cor. II. If a polygon is regular, all the exterior angles are equal and each of them $= \frac{4 \text{ right angles}}{n}$ or $\frac{360 \text{ degrees}}{n}$, n being the number of sides; and each interior angle of the polygon is the supplement of $\frac{360}{n}$ degrees.

EXERCISE XXXV.

1. Find the sum of the angles of (i) a pentagon, (ii) a hexagon, (iii) a heptagon, (iv) a decagon, and (v) a polygon of 20 sides

2. What is the size of each angle of (i) a regular pentagon (ii) a regular hexagon, (iii) a regular octagon, (iv) a regular polygon of 20 sides.

3. Three of the exterior angles of a quadrilateral obtained by producing the sides in order, are respectively 70°, 110°, and 85°. Find the remaining exterior angle, and all the angles of the quadrilateral.

4. Is it possible to have a regular polygon whose exterior angle is 45°? If so, what is the number of sides of this polygon?

5. Is it possible to have regular polygons whose exterior angles are (i) 60°, (ii) 70°, (iii) 12°, (iv) 30°, (v) 18°, (vi) 15°, (vii) 24°, (viii) 100° respectively? Indicate the number of sides in each case where a polygon is possible

6. What is the size of an angle of a regular polygon of 60 sides? [The size of an exterior angle $= 360° \div 60 = 6°$, hence an angle of the polygon $= 180° - 6° = 174°$. Ex. 2 may be worked out in this way].

7. Construct (i) a regular heptagon, each side being 1·5 inches; (ii) a regular decagon, each side being 1 inch. (Use a protractor)

THEOREM 17. [Euclid 1, 26]

95. **If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.**

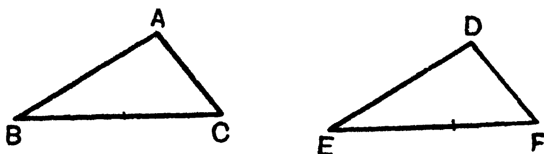


Fig. 211

$\triangle$s ABC and DEF have

$\angle A = \angle D$,

$\angle B = \angle E$,

and side BC = corresponding[1] side EF.

To prove that $\triangle$s ABC, DEF are congruent.

Proof. Because two angles of $\triangle$ABC are equal to two angles of $\triangle$DEF,

∴ the third angle of $\triangle$ABC = the third angle of $\triangle$DEF, [See note 3 (i), § 93]

∴ $\angle C = \angle F$.

Apply $\triangle$ABC to $\triangle$ DEF, so that B falls on E and BC along EF.

Because BC = EF,

∴ C falls on F.

1 Corresponding sides are those which are opposite to equal angles; see Note 1, § 56.

Now $\angle B = \angle E$,

$\therefore$ **BA** falls along **ED**,

Again $\angle C = \angle F$,

$\therefore$ **CA** falls along **FD**,

$\therefore$ the point **A** falls on the point where the lines **ED**, **FD** intersect, i. e., on **D**.

$\therefore$ $\triangle$**ABC** coincides with $\triangle$**DEF**.

$\therefore$ $\triangle$s**ABC**, **DEF** are congruent.

Note. One of the triangles may have to be *turned over* before superposition in order to make them coincide with each other. See **Note** 3, § 56

EXERCISE XXXVI.

1. In Fig 211 state which angles of the $\triangle$**DEF** are equal to $\angle$s **A**, **B**, **C** *respectively* of the $\triangle$**ABC**, and which sides of the $\triangle$**DEF** are equal to the sides **BC**, **CA**, **AB** *respectively* of the $\triangle$**ABC**.

2. **PQR**, **XYZ** are two $\triangle$s (Fig. 212) having **QR**=**YZ**,

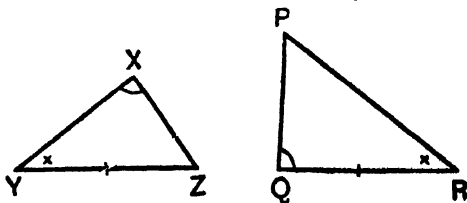


Fig. 212

$\angle$**R**=$\angle$**Y** and $\angle$**Q**=$\angle$**X**; are the triangles congruent? (That is to say, can they be made to coincide with each other by superposition ?)

3. The bisectors of the base angles of an isosceles triangle are equal.

4. The perpendicular drawn from the vertex of an isosceles triangle to the base, bisects the base.

5. The perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides are equal.

6. The perpendiculars drawn from the mid-point of the base of an isosceles triangle to the sides are equal.

7. Through the mid-point of a straight line **AB**, any straight line is drawn. Show that the perpendiculars drawn to it from the points **A** and **B** are equal.

8. *Any point on the bisector of an angle is equidistant from the arms of the angle.*

9. If the diagonal **AC** of a quadrilateral **ABCD** bisects the angles **A** and **C**, show that the quadrilateral has two pairs of equal sides, and that the ∠s **B** and **D** are equal.

10. **ABCD** is a quadrilateral with **AB**=**AD** and **BC**=**CD**. Show that **AC** bisects the ∠s **A** and **C** and that the diagonals are at right angles.

11. **AB** is a straight line terminated by two parallels. Show that the mid-point **O**, of **AB** is equidistant from the parallels.

12. In Ex. 11, if any straight line drawn through **O** cuts the parallels in **P** and **Q**, then **O** is the mid-point of **PQ**.

13. **O** is the mid-point of one of the diagonals of a parallelogram (i. e. a quadrilateral with opposite sides parallel); prove that any straight line drawn through **O** and terminated by two opposite sides of the parallelogram is also bisected at **O**.

14. If the bisector of an angle of a triangle is perpendicular to the opposite side, the triangle is isosceles

15. If the bisector of an angle of a triangle bisects the opposite side, the triangle is isosceles. [Let the bisector of the ∠**A** of the △**ABC** meet the side **BC** in **D**. Produce **AD** to **E**, making **DE**= **AD**. Join **BE**.]

16. **How to find the breadth of a river.** Select a spot where the river is very nearly straight and note a tree **T** (or any other stationary object) on the edge of the other bank. Take your stand at the point **A** near the edge of the river, so that you may see the tree lying *just opposite* to you; then proceed along the bank (at right angles to **AT**) a distance **AB**, counting the number of paces you take in walking from **A** to **B**. Push a stick into the ground at **B**, next walk on again from **B** in the same direction along the bank, and stop at a point **C** as soon as you have taken as many paces from **B** as you took in coming from **A** to **B**. This makes **AB = BC** From **C** walk in a direction at right angles to **AC**, until you come to a point **D** from where you see the tree lying in a straight line with the stick. The distance **CD** measures the breadth of the river. You can get this distance by counting the number of paces from **C** to **D**. To prove that **CD** is equal to **AT** you need only to show that the $\triangle$s **ABT** and **CBD** are congruent

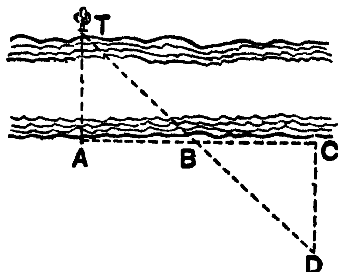


Fig. 213

THEOREM 18.

96. If two right-angled triangles have their hypotenuses equal, and one other side of the one equal to one other side of the other, the triangles are congruent.

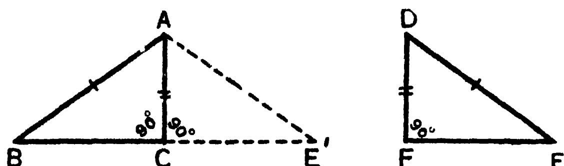


Fig. 214

ABC, DEF are two triangles having ∠s **C, F** right angles, **AB = DE**, and **AC = DF**.

To prove that △s **ABC, DEF** are congruent.

Proof. Apply △ **DEF** to △ **ABC**, so that **D** falls on **A, DF** along **AC**, and the point **E** on the side of **AC** opposite to **B**.

Since **DF = AC, F** will fall on **C**.

Let **E′** be the point where **E** falls.

Since ∠s **ACB, ACE′** (which is the new position of the ∠**DEF**), are right angles,

∴ **BCE′** is one straight line. (*Theor.* 2)

∴ **ABE′** is a triangle.

Since in the △ **ABE′**, **AB = AE′** (**AE′** being the new position of **DE**),

∴ ∠**E′** = ∠**B**. (*Theor.* 5)

Now in the △^{s} ABC, AE′C,

$$\begin{cases} \angle B = \angle E' & \text{Proved.} \\ \angle ACB = ACE' & \text{(being right angles.)} \\ AB = AE', & \end{cases}$$

∴ △^{s} ABC, AE′C are congruent; *(Theo.* 17)

that is, the △^{s} ABC, DEF are congruent.

EXERCISE XXXVII.

1. **ABCD** is a square; **E** is the mid-point of **AB**, and two points **P** and **Q** are taken on **AD**, **BC** respectively, so that **EP**= **EQ** Show that the △s **AEP**, **BEQ** are congruent.

2. If the perpendiculars drawn from two vertices of a triangle to the opposite sides are equal, the triangle is isosceles.

3. If the perpendiculars drawn from the mid-point of one side of a triangle to the other two sides are equal. the triangle is isosceles

4 *If the perpendiculars drawn from a point* (**P**) *upon the arms* (**OA, OB**) *of an angle* **AOB** *are equal then the point* (**P**) *lies on the bisector of the angle* (**AOB**). [See Fig. 215].

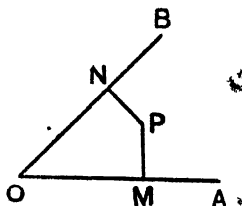


Fig. 215

5. In a quadrilateral **PQRS**, ∠s **P** and **Q** are right angles, and the diagonals **PR**, **QS** are equal, show that **PS**=**QR**.

6. The perpendicular drawn from the centre of a circle on any chord, bisects that chord.

7. **AB** is a chord of a circle whose centre is **O**. Prove that the straight line drawn from **O**, perpendicular to **AB**, will bisect any chord parallel to **AB**.

8. Prove that the perpendiculars drawn from the centre of a circle upon two equal chords are equal.

97. Parallelograms. Various types of quadrilaterals.

A quadrilateral with its opposite sides parallel is called a **parallelogram**.

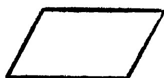


Fig. 216

A **rectangle** is a parallelogram with one of its angles a right angle. [It will be shown later that all the angles of a rectangle are right angles].

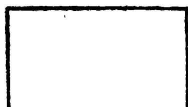


Fig. 217

A **square** is a rectangle with two adjacent sides equal. [It will be shown later that all the sides of a square are equal and all its angles right angles].

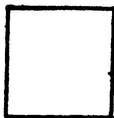


Fig. 218

A **rhombus** is a parallelogram with two adjacent sides equal. [It will be shown later that all the sides of a rhombus are equal].

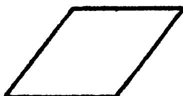


Fig. 219

A **trapezium** is a quadrilateral with one pair of opposite sides parallel, and the other pair not parallel.



Fig. 220

A trapezium in which the non-parallel sides are equal is called an **isosceles trapezium.**

The student is advised to repeat Exercise XXX (Exs. 1—15).

THEOREM 19. [Euclid 1, 33].

98. The straight lines which join the extremities of two equal and parallel straight lines ***towards the same parts***[1] **are themselves equal and parallel.**

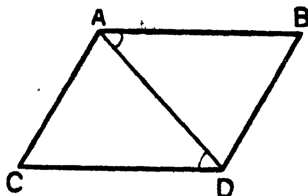


Fig. 221

AB, CD are two equal and parallel straight lines.

To prove that **AC, BD** are equal and parallel.

Join **AD.**

Proof. Since **AB, CD** are parallel, and **AD** meets them

∴ ∠**BAD** = alternate ∠**CDA.** (*Theor.* 14)

Now in the △s **BAD, CDA.**

{ **AB** = **DC** *Given.*
{ **AD** is common,
{ ∠**BAD** = ∠**CDA,** *Proved.*

∴ △s **BAD, CDA** are congruent; (*Theor.* 4)

so that **AC** = **BD,**

and ∠**ADB** = ∠**DAC.**

Since **AD** meets **AC, BD** making the alternate angles **ADB, DAC** equal,

1. The phrase '*towards the same parts*' is explained in the note appended to this theorem.

∴ **AC** and **BD** are parallel. (*Theor.* 13A)

It is thus proved that AC, BD are equal and parallel.

Note. If **A** were joined to **D** and **B** to **C**, the extremities of the lines **AB, CD** would *not* be said to be joined towards the same parts. A glance at the figure (Fig. 221) shows that **AD, BC** intersect and cannot be parallel.

Theorem 19 might also be enunciated as follows :

If a quadrilateral has one pair of opposite sides equal and parallel, then the other pair of opposite sides are also equal and parallel.

THEOREM 20. [(i) and (ii)—Euclid 1, 34].

99. (i) **The opposite angles of a parallelogram are equal,**

(ii) the opposite sides of a parallelogram are equal,

(iii) each diagonal bisects the parallelogram, and (iv) the diagonals of a parallelogram bisect one another.

(i) The opposite angles of a parallelogram are equal.

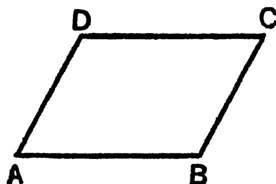


Fig. 222

ABCD is a parallelogram.

To prove that ∠**A** = ∠**C** and ∠**B** = ∠**D**.

Proof. Since **AD** meets the parallels **AB, DC**.

∴ ∠**A** + ∠**D** = 2 right angles (*Theor.* 14)

Again since **DC** meets the parallels **AD, BC**,

∴ ∠**C** + ∠**D** = 2 right angles.

∴ ∠**A** + ∠**D** = ∠**C** + ∠**D**.

∴ ∠**A** = ∠**C**.

In a similar way, it may be proved that

$$\angle B = \angle D.$$

(ii) and (iii) The opposite sides of a parallelogram are equal; each diagonal bisects the parallelogram.

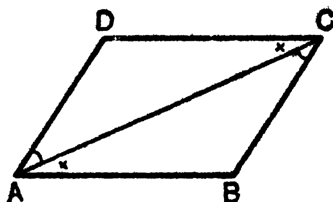


Fig. 223

ABCD is a parallelogram, and AC one of its diagonals.

To prove that AB = DC, AD = BC, and AC bisects the parallelogram.

Proof. Since AC meets the parallels AB, CD,

∴ ∠BAC = alt. ∠DCA. (*Theor.* 14)

Again since AC meets the parallels AD, BC,

∴ ∠ACB = alt. ∠CAD.

Now in the △ˢ ABC, CDA,

$$\begin{cases} \angle BAC = \angle DCA, & \textit{Proved.} \\ \angle ACB = \angle CAD, & \\ \text{AC is common,} & \end{cases}$$

∴ the △ˢ ABC, CDA are congruent. (*Theor.* 17)

∴ AB = CD, and BC = AD;

and since the △ˢ ABC, CDA are equal in all respects,

∴ the diagonal AC divides the parallelogram into two halves, i.e., bisects the parallelogram.

Similarly it may be shown that the other diagonal, **BD**, also bisects the parallelogram.

(iv) The diagonals of a parallelogram bisect one another.

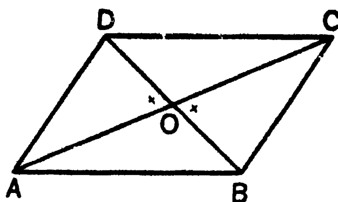


Fig. 224

ABCD is a parallelogram, and the diagonals **AC, BD** intersect at **O.**

To prove that AO = OC, BO = OD.

Proof. Since **AC** meets the ||ˢ **AD, BC,**

∴ ∠OAD = alt. ∠OCB.

Now in the △ˢ **AOD, COB,**

$\left\{\begin{array}{l} \angle \text{OAD} = \angle \text{OCB}, \qquad \textit{Proved.} \\ \angle \text{AOD} = \text{vert. opp. } \angle \text{COB}; \\ \text{AD} = \text{CB. [Theor. 20 (ii), already proved].} \end{array}\right.$

∴ △ˢ **AOD, COB** are congruent ;

∴ **OA = OC** and **OD = OB.**

100. Corollaries to Theorem 20.

Cor. I. *If two lines are parallel, all points on either of the lines are at the same perpendicular distance from the other.*

Cor. II. *If a parallelogram has one of its angles a right angle, all its angles are right angles.*

Thus all the angles of a rectangle or a square are right angles.

Cor. III. *If two adjacent sides of a parallelogram are equal, all its sides are equal.*

Thus all the sides of a square or a rhombus are equal.

Cor. IV. *If one angle of a parallelogram is not a right angle, then none of its angles are right angles.* [A parallelogram which is not a rectangle is often called an *oblique* parallelogram].

The proofs of the above corollaries are left to the student.

EXERCISE XXXVIII.

1. *An equilateral 4-sided figure is a rhombus or a square*

2. *An equiangular 4-sided figure is a rectangle*

3. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

4. *If the opposite angles of a quadrilateral are equal the figure is a parallelogram*

5. *Prove that the diagonals of a rhombus bisect one another at right angles*

6. Prove that the diagonals of a rectangle are equal

7. Prove that the diagonals of a square are equal and at right angles to each other

8. If the diagonals of a quadrilateral bisect one another, show that the quadrilateral is a parallelogram.

9. If the diagonals of a quadrilateral bisect one another at right angles, the quadrilateral is a rhombus or a square.

10. *If the diagonals of a parallelogram are equal the figure is a rectangle.*

11. *If the diagonals of a parallelogram are equal and at right angles to one another the figure is a square.*

12. *If the diagonals of a parallelogram are unequal the figure cannot be a rectangle.*

13. Each of the diagonals of a rhombus, or a square, bisects the angles which it joins.

14. *If a parallelogram has not all its sides equal, neither of the diagonals bisects the angles which it joins*

15. The bisectors of two opposite angles of a parallelogram whose sides are not all equal are parallel.

16. The bisectors of two adjacent angles of a parallelogram are at right angles to each other.

17. Prove that the bisectors of the angles of a parallelogram whose sides are not all equal, form a rectangle.

18. Any straight line drawn through the intersection, **O**, of the diagonals of a parallelogram, and terminated by a pair of opposite sides is bisected at **O**.

19. The straight line which joins the mid-points of two opposite sides of a parallelogram is parallel to the other sides.

20. **AB** is a straight line and from two points **M** and **N** in it, perpendiculars **MP**, **NQ** are drawn to **AB** in the same direction. If **MP**=**NQ**, show that **PQ** is parallel to **AB**.

21. From several points **L**, **M**, **N** on a line **AB**, perpendiculars **LP**, **MQ**, **NR** are drawn to **AB** in the same direction. If **LP**=**MQ**=**NR**=.., then the points, **P**, **Q**, **R**...are collinear and lie on a straight line parallel to **AB**.

22. **PQRS**, and **PQR'S'** are two parallelograms standing on the same base **PQ**. Show that **RR'S'S** is a parallelogram.

23. **A pair of parallel rulers.** **A** and **B** are two rulers (Fig 225); **P** and **Q** are two points on **B**, such that **PQ** is parallel to its edges. **R**, **S** are two points on the other ruler **A**, such that **RS** is parallel to its edges, and the distance **RS**=distance **PQ**. Two

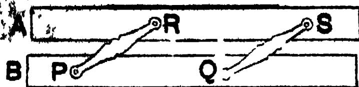


Fig. 225

equal rods **PR** and **QS** are hinged to these points as shown in the figure. Now hold the ruler **B** firmly, move the other (**A**) about, and rule a number of lines against it. Prove that these lines are parallel.

(*b*)

(on axes of symmetry, See § 57, Note 4)

24. Show that a rhombus is symmetrical about each of its diagonals [In other words, you are to show that if a rhombus is folded about a diagonal, the triangles on opposite sides of it coincide].

25. Can a parallelogram which is not equilateral, be symmetrical about a diagonal? [**ABCD** is such a parallelogram (Fig. 226); the Δ^s **ABC, ADC** are congruent, but if you fold about the diagonal **AC**, the Δ**ADC** falls into the position **AD′C**, and evidently does not coincide with Δ **ABC**, see Ex. 14].

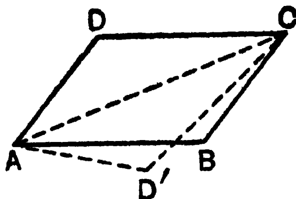
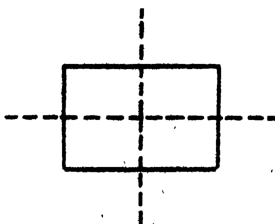


Fig. 226


Fig. 227

26. Name the axes of symmetry of a rectangle which is not a square. [Evidently the lines joining the mid-points of opposite sides (Fig. 227) are axes of symmetry].

27. Name all the axes of symmetry of a square.

28. Does an oblique parallelogram possess any axis of symmetry?

29. If the diagonal **AC** bisects the angles **A, C** of a quadrilateral **ABCD**, show that **AC** is an axis of symmetry.

30. Name the axis of symmetry of an isosceles trapezium. [see § 97].

THEOREM 21.

101. **If there are three or more parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, the corresponding intercepts on any other straight line that cuts them are also equal.**

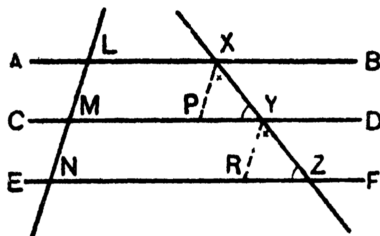


Fig 228

The parallels **AB, CD, EF** are cut by the line **LMN** and the intercepts **LM** and **MN** are equal. Let any other straight line **XYZ** cut the parallels in **X, Y, Z**.

To prove that the corresponding intercepts **XY, YZ** are equal.

Through **X, Y** draw **XP, YR** parallel to **LN.**

Proof. Since **LP** is a parallelogram, ∴ **XP = LM.**
(*Theor.* 20)

Since **MR** is a parallelogram, **YR = MN.**

But **LM = MN.** *Given.*

∴ **XP = YR.**

Since **XP** and **YR** are **parallel** (being each parallel to **LN**) and **XZ** meets them,

∴ **∠PXY** = corresp. **∠RYZ.**

Again since **XZ** meets the parallels **CD, EF,**

∴ ∠**XYP** = corresp. ∠**YZR,**

Now in the △ˢ **XPY, YRZ,**

$$\begin{cases} \angle PXY = \angle RYZ, & \textit{Proved.} \\ \angle XYP = \angle YZR, & \text{,,} \\ XP = YR, & \text{,,} \end{cases}$$

∴ the △ˢ are congruent.

∴ **XY** = **YZ.**

102. Applications of Theorem 21.

(a) The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.

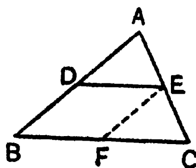


Fig. 229

Through the mid-point **D** of the side **AB** (Fig. 229) of the △ **ABC, DE** is drawn parallel to the side **BC,** meeting the third side **AC** in **E.**

To prove that **E** is the mid-point of **AC.**

The result is deduced immediately from Theorem 21 by imagining a line to be drawn through **A,** parallel to **BC.**

A direct proof is however given below.

Proof. Through **E** draw **EF** parallel to **AB.**

Since **DF** is a parallelogram,

EF = **BD.**

But **BD** = **AD.** *Given.*

∴ **EF** = **AD.**

Since **AC** meets parallels **AB, EF,**

∴ ∠**DAE** = corresp. ∠**FEC.**

Since **AC** meets the parallels **DE, BC,**

$\angle$**AED** = corresp. $\angle$**ECF.**

Now in the $\triangle^s$ **AED, ECF**

$\left\{\begin{array}{l}\angle DAE = \angle FEC, \\ \angle AED = \angle ECF. \\ AD = EF,\end{array}\right.$

$\therefore$ the $\triangle^s$ are congruent,

$\therefore$ **AE** = **EC.**

It will not be out of place to consider here the allied theorem.

(b) The straight line joining the mid-points of two sides of a triangle is parallel to the third side and is equal to half of it.

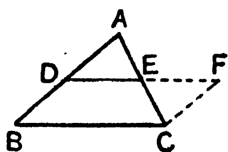


Fig. 230

D, E are the mid-points of the sides **AB, AC** of the $\triangle$ **ABC** (Fig. 230). Then **DE** is parallel to **BC,** and is equal to $\frac{1}{2}$ **BC.**

Through **C** draw **CF** $\parallel$ to **BA,** meeting **DE** produced in **F.**

In the $\triangle^s$ **AED, CEF.**

$\left\{\begin{array}{l}AE = EC, \quad \textit{Given.} \\ \angle AED = \text{vert. opp. } \angle CEF, \\ \angle ADE = \text{alt. } \angle CFE.\end{array}\right.$ ($\because$ **CF** is $\parallel$ to **BA** and **DF** meets them),

$\therefore$ the $\triangle^s$ are cougruent,

$\therefore$ **AD** = **CF.**

But **AD** = **BD**

$\therefore$ **BD** = **CF.**

And **BD** is also parallel to **CF.**

∴ **DF** is equal and parallel to **BC.** (*Theor.* 19)

Now from the congruent △ˢ **AED, CEF,**

DE = EF.

∴ **DE** = $\frac{1}{2}$ **DF** = $\frac{1}{2}$ **BC.**

Thus **DE** is parallel to **BC,** and equal to $\frac{1}{2}$ **BC.**

Problem 7.

(c) To divide a given straight line into any number of equal parts.

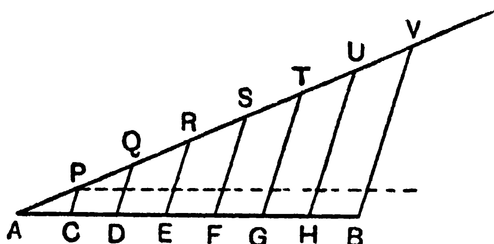


Fig. 231

Suppose it is required to divide the straight line **AB** into 7 equal parts.

Through **A** draw a line making any convenient angle with **AB**, and mark a point **P** in it adjacent to **A**; cut off parts **PQ, QR, RS, ST, TU, UV** each equal to **AP**. Thus **AV** is divided into 7 equal parts at the points **P, Q, R, S, T, U.**

Join **BV**, and through **P, Q, R, S, T, U** draw straight lines parallel to **BV**, cutting **AB** in the points **C, D, E, F, G, H.** Since the intercepts **AP, PQ, QR, RS, ST, TU, UV** are all equal,

∴ the corresponding intercepts **AC, CD, DE, EF, FG, GH, HB** are all equal. [*Theorem* 21]

Thus **AB** is divided into 7 equal parts.

Note. The method is quite general. It should be observed that **AC** $=\frac{1}{7}$ **AB**, **AD** $=\frac{2}{7}$ **AB**, **AE** $=\frac{3}{7}$ **AB**,......Thus this method enables us to cut off from any line a part which shall be a given fraction of it.

(d) **The Diagonal Scale.** In Fig. 231, if you imagine straight lines to be drawn through **P, Q, R, S, T, U**, all parallel to **AB**, these lines will divide **VB** into 7 equal parts (Theor. 21), and each of these parts will be equal to **PC**. Thus **VB** $=7$ **PC**. Similarly **UH** $=6$ **PC**, **TG** $=5$ **PC**, **SF** $=4$ **PC**, **RE** $=3$ **PC**, **QD** $=2$ **PC**. We may put this result in another form :—**PC** $=\frac{1}{7}$ **VB**, **QD** $=\frac{2}{7}$ **VB**,

RE $=\frac{3}{7}$ **VB**, **SF** $=\frac{4}{7}$ **VB**, **TG** $=\frac{5}{7}$ **VB**, **UH** $=\frac{6}{7}$ **VB**.

This property is used in the construction of what is called the **Diagonal Scale** (already referred to in the note on page 24) for measuring hundredths of an inch.

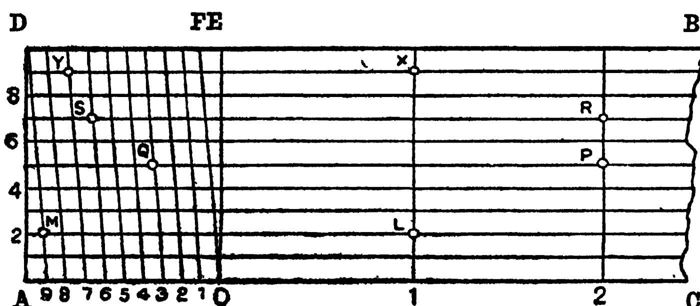


Fig. 232

It is in the form of a rectangle as shown in Fig. 232. The larger divisions of the bottom line are inches, of these the one on the left is subdivided into tenths of an inch. The edge **AD** (which may be of any length) is divided into ten equal parts, and through the points of division, parallels are drawn to **AC** or **DB**. The top line **DB** is divided in the same way as **AC**, and the diagonal scale is obtained by joining diagonally the divisions of the bottom line to those of the top line as shown in the figure. In practice, however, these lines are drawn, by first joining the top left-hand

corner **D** to division 9 of the bottom line and then drawing parallels to this line, through the other points of division, viz., 0, 1, 2, etc. We have thus a network of small parallelograms, each of which has its top and bottom sides equal to a tenth of an inch. The intercepts made by the diagonal lines upon any of the parallels to **AC** are all equal, each being a tenth of an inch. The intercept **FE** made on the top line **BD** by the diagonal line **OF** and the perpendicular **OE** (⊥ to **AC**) is also a tenth of an inch. The intercepts made by these two lines (**OF**, **OE**) on the other parallels to **AC**, beginning from the first, (i.e., the lowest) are respectively $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$,...etc. of **EF** i.e., $\frac{1}{100}$, $\frac{2}{100}$, $\frac{3}{100}$, etc of an inch.

How to use a Diagonal Scale—To set off a certain distance, say 2·35 inches, from the diagonal scale, place the two points of the dividers on *parallel* 5, at the points where this parallel cuts *perpendicular* 2 and *diagonal* 3. These points are marked **P, Q** in Fig. 232. The distance **PQ**, is 2·35 inches and may be carried anywhere with the aid of the dividers.

Ex 1. Read off the distances **XY**, **RS**, **LM**, (Fig. 232).

Ex. 2. Trace two lines on the paper and measure them to the nearest hundredth of an inch, by transferring the lengths to a Diagonal Scale with dividers.

EXERCISE XXXIX.

1. *P, Q, R are the mid-points of the sides BC, CA, AB of a* △*ABC. Show that AQPR, BRQP, CPRQ are all parallelograms.*

2. *Show that the straight lines joining the mid-points of the sides of a triangle divides the triangle into four congruent triangles.*

3. Show that any straight line drawn from a vertex of a triangle to the opposite side is bisected by the line joining the mid-points of the other two sides.

4. Show that the line joining any vertex of a triangle to the mid-point of the opposite side bisects the line joining the mid-points of the other two sides.

5. **ABC is a triangle right-angled at A, and D is the mid-point of the hypotenuse BC. Show that AD=BD=CD.** [Join **D** to the mid-point of **AB**.]

6. *The mid-points of the sides of any quadrilateral are the vertices of a parallelogram.*

7. **ABCD** is a parallelogram, and **E**, **F** are the mid-points of **AB**, **CD**. Show that **EC**, **AF** trisect the diagonal **BD**; and **DE**, **BF** trisect **AC**.

8. *The straight lines joining the mid-points of opposite sides of any quadrilateral bisect one another.* [Use Ex. 6.]

9. **AB** is a straight line and **C** its mid-point. From **A**, **B**, **C**. perpendiculars **AL**, **BM**, **CN** are drawn to any other straight line **PQ**. Show that,

(i) **AL+BM=2CN**, if the points **A** and **B** are on the same side of **PQ**.

(ii) *The difference of* **AL**, **BM=2CN** if **A** and **B** are on opposite sides of **PQ**.

10. Show that the straight line joining the mid-points of the non-parallel sides of a trapezium is parallel to the other two sides, and that its length is half the sum of the lengths of the parallel sides.

11. From the vertices of a parallelogram perpendiculars are drawn to any straight line lying outside the parallelogram. Show that the sum of the perpendiculars from one pair of opposite vertices is equal to the sum of those from the other pair.

12. The sum of the distances of any point on the base of an isosceles △, from the equal sides is equal to the distance of an extremity of the base from the side opposite to it.

13. **P** is any point on the base **BC** of an isosceles triangle **ABC**, and **PM**, **PN** are drawn perpendicular to **AB**, **AC**. Show that **PM+PN** is constant (i.e. has the same value wherever the point **P** is taken on the base).

14. **O** is any point within an equilateral triangle show that the sum of the distances of **O** from the sides of the △ has the same

value wherever the point O is taken ; in other words, the sum of the distances is constant. [Show that the sum of the distances = distance of any vertex of the △ from the opposite side ; this result may be obtained by drawing through O a parallel to one of the sides of the △ and using Ex. 12].

15. **Given the mid-points of the sides of a triangle to construct the triangle.**

EXERCISE XL.

1. Trisect a given line.

2. Draw any line **AB** and divide it into 5 equal parts.

3. The distance **OA** represents an interval of 10 seconds on



Fig. 233

a certain scale Mark points A_1, A_2, A_3..., A_9 on **OA** so that OA_1, OA_2, OA_3..., OA_9, may represent intervals of 1 second, 2 secs, 3 secs,...9 secs., respectively.

4. Take a straight line 4 in long and divide it into 7 equal parts ; measure each of these parts with a Diagonal Scale correct to the nearest hundredth of an inch, and verify by calculation.

5. Draw any straight line, and cut off a part equal to $\frac{5}{11}$ of the whole line.

6. Construct lines which shall be (i) $\frac{4}{5}$, (ii) $\frac{3}{8}$ of a given line.

7. Draw a line 1·5 inches long and construct a line which shall be 2·5 times as long. Measure the line you construct and verify by calculation.

8. **To divide a given line *internally* in a given ratio.** For example, to divide the line **AB** internally in the ratio 2 : 3, i.e., to find a point **C** in **AB** (between **A** and **B**) such that **AC** : **CB** = 2 : 3 [Divide **AB** into 2+3 i.e. 5 equal parts (Prob.7): **AC** $=\frac{2}{5}$**AB**, **CB** $=\frac{3}{5}$ **AB** so that **AC** : **CB** $=\frac{2}{5}$ **AB** : $\frac{3}{5}$ **AB** $=\frac{2}{5}$: $\frac{3}{5}$ = 2 : 3].

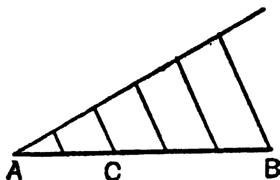


Fig. 234

9. Divide a line **CD** *internally* in the ratio 3 : 7.

10. Divide a line **PQ** internally in the ratio 4 : 5.

11. **To divide a given line *externally* in a given ratio.**

For example, to divide the line **AB** externally in the ratio 7 : 2, i. e., to find a point **C** in **AB** *produced* such that **AC** : **BC** = 7 : 2, [Through **A** draw a line making any convenient angle with **AB**; mark a point **P** in this line and cut off **PQ**, **QR**, **RS**, **ST**, **TU**, **UX**, all equal to **AP** Thus **AX** is divided into 7 equal parts. Join the *second* division from **X**, i.e **T**, to **B**, and draw parallels to **TB** through **P**, **Q**, **R**, **S**, **U**, **X**.

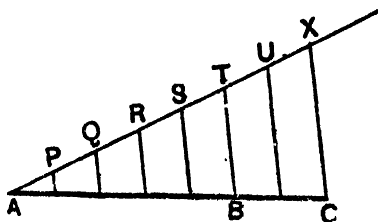


Fig. 235

The parallel through **X** cuts **AB** in the required point **C**.

For these parallels divide **AC** into 7 equal parts, and **BC** is 2 such parts, so that **AC** : **BC** = 7 : 2. [Another method—Divide **AB** into (7-2) i. e., 5 equal parts and produce **AB** to **C** such that **BC** = 2 such parts.]

12. Divide the line **AB** [Ex. 11.] *externally* in the ratio 2 : 7 [The point of section **C** must lie in **BA** produced in this case, for **BC** > **AC**].

13. Alternative construction for Problem 7.

To divide **AB** into 5 equal parts.

Through **A** draw **AP** at any convenient angle with **AB**, and mark off on it *four* equal parts **AX**, **XY**, **YZ**, **ZU** of any length (Fig. 236). Through **B** draw **BQ** ‖ to **AP** and mark off on it four parts **BU′**, **U′Z′**, **Z′Y′**, **Y′X′**, equal to the parts taken on **AP.** Join **XX′**, **YY′** **ZZ′** **UU′**. These lines cut **AB** into 5 equal parts. Prove the result.

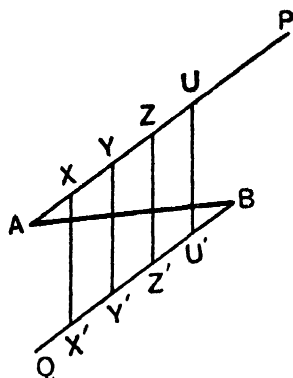


Fig. 236

14. Take a stiff piece of paper, and prepare a diagonal scale with it for measuring up to hundedths of an inch (The total length of the scale to be 6 inches)

CHAPTER VII.

CONSTRUCTION OF TRIANGLES AND QUADRILATERALS.

103. The construction of triangles and quadrilaterals has already been discussed in Chapter II (§ § 41, 42). In the present Chapter we shall treat the object in a more theoretical way.

To construct a triangle which shall have its three sides equal to three given straight lines.

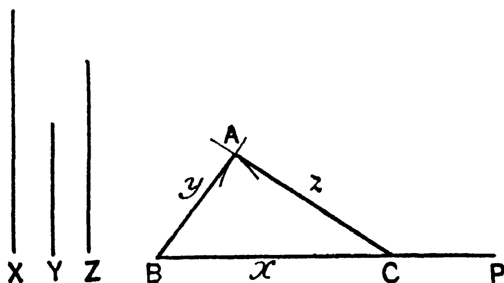


Fig. 237

X, Y, Z are three given straight lines.

To construct a triangle whose sides shall be equal to **X, Y, Z,** respectively.

Construction. Draw any straight line **BP** and cut off from it **BC = X.**

With centre **B** and radius = **Y** draw a circle.

With centre **C** and radius = **Z** draw a circle.

Let the circles intersect at **A**.

Join **AB**, **AC**.

Then **ABC** is the required triangle, for by construction the sides **BC**, **CA**, **AB** are respectively equal to **X**, **Z**, **Y**.

Note—For the construction to be possible it is necessary that any two of the given sides should be together greater than the third side (Theorem 11); for if $Y+Z$ were less than **X**, the circles drawn with centres **B** and **C** and radii equal to **Y**, **Z** respectively would not intersect. These circles which intersect at **A** will also intersect at another point on the other side of **BC**. Thus there are two triangles lying on opposite sides of **BC**.

It is convenient to draw the longest side first.

EXERCISE XLI.

1. Draw any triangle **ABC**, and construct another triangle **PQR** whose sides shall be equal to those of the $\triangle$**ABC**, [These two triangles are congruent. (Theorem 7)].

2. Draw a straight line **AB**, and construct an equilateral triangle on it.

3. Draw a straight line **BC** and construct an isosceles triangle **ABC** on **BC** as base so that **AB** = **AC** = 2**BC**.

4. Make an angle of 60° (without using a protractor or a set-square.)

5. Make an angle of 120° (without using a protractor or a set-square).

6. Make an angle of 30° (with ruler and compasses only).

7. Make an angle of 15° (with ruler and compasses only).

8. To trisect a right angle XOY traced on paper.

Draw an equilateral Δ**AOB** with one side lying along one of the arms (**OX**) of the given right angle as shown in Fig. 238; then $\angle$**AOB**$=60°$, and the $\angle$**BOY**$=90°-60°=30°$. Bisect the $\angle$ **BOX** by drawing a line **OC** through **O.** The lines **OB**, **OC** trisect the given right angle **XOY** for the $\angle$s **AOC**, **COB**, **BOY** are equal being each $=30°$.

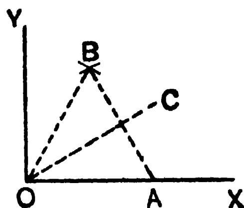


Fig. 238

104. The construction of triangles with *two sides and the included angle given, one side and the angles at the extremities of this side given or two sides and the angle opposite to one of them given,* have already been considered in Exercise XI, (Exs. 6, 9, 12), and need not be repeated here. There the angles and sides were assigned numerically. If the three assigned parts are given as lines or angles *traced on paper*, the constructions can be effected with ruler and compasses.

The student must carefully repeat Exercise XI.

(*a*) *Three sides given* :—Exs. 1--4

(*b*) *Two sides and the included angle given* :—Exs. 6—8.

(*c*) *One side and the angles at its extremities given* :—Exs. 9, 10, 11.

(*d*) *Two sides and the angle opposite to one of them given* :—Ex. 12.

It may be noted that if one side and two angles, one of which is opposite to this side, be given, the construction of the triangle may be reduced to case (c) by Theorem 16 (see § 93. Note 3). Suppose it is

required to construct a triangle **ABC**, given **BC** and $\angle^s$ **B** and **A**.

Now $\angle A + \angle B + \angle C = 180°$

$\therefore \quad \angle C = 180° - (\angle A + \angle B)$.

Thus $\angle C$ is known.

Knowing **BC** and $\angle^s$ **B** and **C**, the triangle may be constructed as in case (*c*).

The triangle may also be constructed directly from the data.

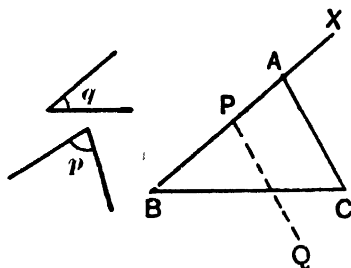


Fig. 239

Let **BC** be the given side. Suppose the angles **A**, **B** are to be equal to two given $\angle^s$ *p*, *q*, respectively.

From **B** draw **BX**, making $\angle$**XBC** = given $\angle q$ (Fig. 239)

Take any point **P** in **BX**, and from **P** draw **PQ** making $\angle$**BPQ** = given $\angle p$.

Through **C** draw a parallel to **PQ** cutting **BX** in **A**, then **ABC** is the required triangle. The proof is left to the student.

Thus if **three** parts of a triangle are assigned in any of the following ways, the triangle can be definitely constructed :—

(1) Three sides.

(2) Two angles and one side.

(3) **Two sides and one angle** [this gives rise to two subcases according as the given angle is the angle (i) *included by the two sides*, or (ii) *opposite to one of the given sides.*]

It should be observed that in each of these cases an infinite number of triangles can be constructed with the given parts. In each of the cases (1), and (2), and subcase (i) of (3), the triangles constructed are congruent, and are all of unique size and shape. Subcase (ii) of (3), is a little more complex, and is known as the **ambiguous** case, and we have seen in Ex. 12, p. 85, how with the given parts sometimes no triangle may be possible, sometimes only one set of triangles of unique size and shape may be possible, sometimes two sets of triangles of distinct shapes and sizes may be possible. In subcase (i) of (3) a triangle is possible with the given parts if the given angle is less than two right angles, in case (1) a triangle is possible only when any two of the given sides is greater than the third; in case (2) a triangle is possible only when the sum of the given angles is less than two right angles. Regarding the **ambiguous case** the student is advised to go again very carefully through Ex. 12, p. 85.

One case has not been considered yet,—**the case of three angles being given.** In this case an infinite number of triangles are possible. They are all of the same shape, but may be of different sizes as shown in Fig. 240.

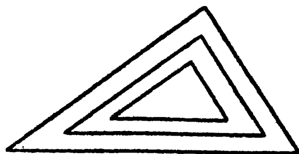


Fig. 240

It should be noted that no triangle is possible if the sum of the three given angles is not exactly two right angles.

105. To construct a right-angled triangle having given the hypotenuse and one side.

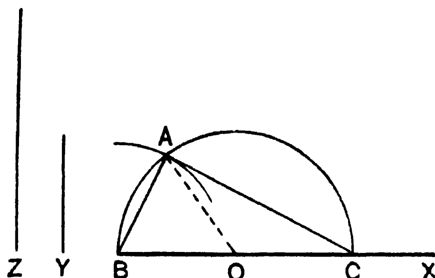


Fig. 241

To construct a right-angled triangle whose hypotenuse shall be equal to **Z**, and one side equal to **Y**.

Construction. Take any line **BX**, and cut off **BC** = **Z**.

Bisect **BC** at **O**, and with centre **O** and radius **OB** describe a circle, this circle will pass through **C** (∵ **OB** = **OC**).

With centre **B** and radius = **Y** describe a circle, cutting the former circle at **A**.

Join **AB**, **AC**.

Then **ABC** is the required triangle.

Proof. Join **OA**.

Since **OA** = **OB**,

∴ ∠**OAB** = ∠**OBA**,

Since **OA** = **OC**,

∴ ∠**OAC** = ∠**OCA**,

∴ ∠**OAB** + ∠**OAC** = ∠**OBA** + ∠**OCA**

i.e.. ∠**BAC** = ∠**ABC** + ∠**ACB**.

Thus the $\angle BAC$ of the $\triangle ABC$ is equal to the sum of the other two angles,

$$\therefore \quad \angle BAC = 1 \text{ right angle.}$$

[See § 93, Note 3 (iv).]

Note. If any point **A** on a circle is joined to the extremities of a diameter of the circle, the angle formed at **A** is a right angle, in other words, **a diameter of a circle subtends a right angle at any point on the circumference** (See Ex. 17, p. 201)

This property gives us a method for **drawing a perpendicular to a given straight line AB from a given point O outside it.**

Join **O** to any point **P** in **AB** (Fig. 242.) and bisect **OP** at **C.** With centre **C** and radius **CO** describe a circle. This circle passes through **P** ($\because$ **CP** = **OC**) and cuts **AB** in another point **N.** Join **ON.**

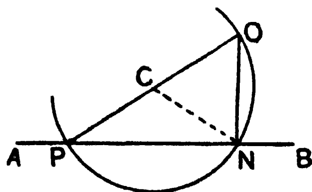


Fig. 242

Then **ON** is perpendicular to **AB.**

The proof is left to the student.

Ex. 1. **AB** is a fixed straight line and **O** a fixed point outside **AB** If **P** be a variable point on **AB**, show that the circle described an **OP** as diameter cuts **AB** again in a fixed point.

Ex. 2. Construct a right-angled triangle with the hypotenuse = 3 inches, and a side = 2 inches. Measure the other side and the acute angles.

Ex. 3. Construct a right-angled triangle with the sides measuring 3 inches and 4 inches respectively. Measure the hypotenuse. Do you find it to be $\sqrt{3^2+4^2}$ inches ?

Ex. 4. Given the hypotenuse and one of the acute angles. Construct the right-angled triangle.

Ex. 5. Given a side and the acute angle opposite to it; construct the right-angled triangle.

106. Construction of triangles from other data.

We shall take an example by way of illustration.

To construct a triangle having given the perimeter (i. e., the total length of the three sides) **and two angles.**

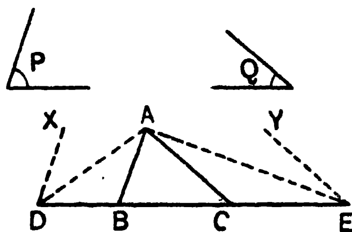


Fig. 243

To get a clue to the construction,[1] we shall *analyse* as follows. Suppose **ABC** to be the required triangle with the ∠s **ABC** and **ACB** equal to the given ∠s **P** and **Q** respectively (Fig. 243). Produce **BC** both ways to **D** and **E**, making **BD** = **BA**, and **CE** = **CA**. Then **DE** is equal to the perimeter of the △**ABC**. Join **AD**, **AE**.

Since **BA** = **BD**, ∠**BAD** = ∠**BDA**.

Now ∠**ABC** = the sum of the int. opp. ∠s **BAD**, **BDA** = 2∠**BDA**

∴ ∠**BDA** = $\frac{1}{2}$ ∠**ABC** = $\frac{1}{2}$∠*P*,

Similarly, ∠**CEA** = $\frac{1}{2}$∠**ACB** = $\frac{1}{2}$ ∠*Q*.

This analysis suggests the following method for the construction of the triangle :

---

1. When it is difficult to find out a method of construction directly, a clue may often be obtained by assuming the problem solved, and then reasoning backwards. This process is known as **Analysis**.

Take a line DE equal to the given perimeter (Fig. 243). At D and E make ∠s EDX and DEY, equal respectively to ∠s *P* and *Q*. Bisect the ∠s EDX and DEY, and let the bisectors meet at the point A. Through A draw AB, AC parallel to XD and YE respectively, meeting DE at B and C. Then ABC is the required triangle.

The proof is left to the student.

EXERCISE XLII.

1. Construct a triangle **ABC**, such that ∠**A** = 60°, ∠**B** = 45°, and **AC** = 2 inches (you are not to use a protractor).

2. Construct a triangle **ABC** such that **AB** = 3 inches, ∠**A** = 45°, **BC** = 2·6 inches, and ∠**C** acute. Construct another triangle **A'B'C'** such that **A'B'** = 3 inches, ∠**A'** = 45°, **B'C'** = 2·6 inches, and ∠**C'** acute. Are the two triangles congruent ?

3. Construct a triangle **ABC** such that **AB** = 3 inches, ∠**A** = 45°, **BC** = 2·6 inches, and ∠**C** obtuse. Construct another triangle **A'B'C'** such that **A'B'** = **AB**, ∠**A'** = ∠**A**, **B'C'** = **BC**, and ∠**C'** acute. Are the △s **ABC**, **A'B'C'** congruent ?

4. Draw an isosceles triangle **ABC**, such that the base **BC** = 3 inches, and the perpendicular drawn from the vertex **A** to the base **BC** = 2 inches [Give a proof].

5. Draw an isosceles triangle so that each of the equal sides may be 3 inches, and the perpendicular from the vertex to the base may be 2·5 inches.

6. Draw an isosceles triangle with the vertical angle equal to a given angle, and the perpendicular from the vertex upon the base equal to a given straight line. [Give a proof].

7. Draw an equilateral triangle, so that the perpendicular drawn from any vertex to the opposite side may be equal to a given straight line. [Give a proof].

8. Construct a $\triangle$ **ABC**, such that the angles **B** and **C** may be 40° and 60° respectively, and the perpendicular **AD** drawn from **A** to **BC** may be 2 inches. [Use a protractor].

9. Construct a $\triangle$ **ABC** given the lengths of **AB, AC,** and the perpendicular from **A** upon **BC**.

10. Construct a right-angled triangle having given the hypotenuse and the perpendicular drawn to it from the opposite vertex.

11. Construct an isosceles triangle with a given base and a given vertical angle

12 Construct a triangle **ABC**, so that **AB** $=3$ in., **BC** $=3{\cdot}5$ in., and the perpendicular from **A** upon **BC** $=2$ inches. How many distinct triangles are possible ? [Give a proof].

13. **P** is a given point, **Q** is a given point in a given straight line. Show how to find a point **O** on the given line such that **OP**$+$**OQ** may be equal to a given length. When does the problem fail ?

(The problem fails when the given length $<$ **PQ**)

14. Construct a triangle **ABC**, given $\angle$**B**, **BC**, and **AB**$+$**AC** [Analyse by assuming the triangle to be constructed and reasoning backwards, see Note, § 106].

15. Construct a right-angled triangle having given the hypotenuse, and the sum of the sides containing the right angle.

[Let **BD** be equal to the given length of the sum of the two sides. Through **D** draw a line **DE**, making $\angle$**BDE**$=45^\circ$; with centre **B** and radius $=$ the given length of the hypotenuse draw a circle; let this circle cut **DE** in **C**. Through **C** draw **CA** perpendicular to **BD**. Join **BC**. Then **ABC** is the required right-angled triangle. Prove that **AB**$+$**AC**$=$**BD**.]

Fig. 244

16. Construct a triangle **ABC**, given the lengths of **AB**, **BC** and the median from **A**. [Give a proof].

17. Construct a triangle **ABC** right-angled at **A**, with the side **AB** = 1·5 inches, and the median drawn to the hypotenuse equal to 2 inches. [Use the property that in a right-angled triangle the median drawn to the hypotenuse is half the hypotenuse. (See Ex. 5, p. 228)].

18. Construct a right-angled triangle given one of the two sides containing the right angle and the sum of the hypotenuse and the other side. [Analyse]

19. Construct a triangle whose perimeter is 8 inches and two angles are 60° and 70° respectively. [Use a protractor].

20. Construct a triangle **ABC**, having given ∠**B** = 30°, **BC** = 2 inches, and **AB** − **AC** = 0·5 inch. [Analyse].

21. Construct a triangle **ABC**, having given ∠**B** = 30°, **BC** = 2 inches and **AC** − **AB** = 0·5 inch. [Analyse].

22. Construct a triangle **ABC**, given ∠**A**, **BC**, and **AB** + **AC**.

23. Construct a triangle **ABC**, given ∠**A**, **BC**, and **AC** − **AB**.

107. Construction of Quadrilaterals.—An Exercise on the construction of quadrilaterals has already been given in § 42. We proceed to treat the subject in a more systematic manner.

Case 1. To construct a quadrilateral, having given the lengths of the four sides and one angle.

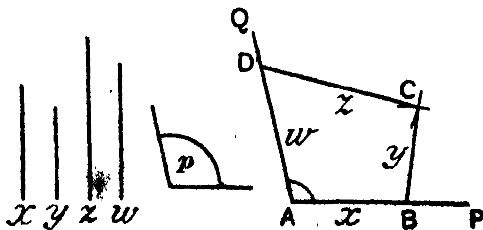


Fig. 245

To construct a quadrilateral **ABCD**, such that **AB** = x, **BC** = y, **CD** = z, **DA** = w, and $\angle$**A** equal to a given angle p.

Construction. Draw a line **AP** and cut off from it **AB** = x. Make $\angle$**BAQ** = given $\angle p$. From **AQ** cut off **AD** = w. With centre **B** and radius = y draw a circle; with centre **D** and radius = z draw another circle cutting the former at **C**. Join **BC**, **DC**. Then **ABCD** is the required quadrilateral.

The proof is left to the student.

Case II. To construct a quadrilateral having given the four sides and a diagonal.

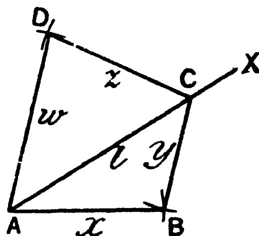


Fig. 246

To construct a quadrilateral **ABCD**, such that **AB**, **BC**, **CD**, **DA** are respectively equal to x, y, z, w, and the diagonal **AC** = l.

Construction. Draw any line **AX** and cut off **AC** = l. With centre **A** and radius = x draw a circle, with centre **C** and radius = y, draw a circle, cutting the former at **B**. In the same way by drawing two other circles with centres **A** and **C**, and radii respectively equal to w and z, you get the point **D**. Join **AB**, **BC**, and **AD**, **DC**. Then **ABCD** is the required quadrilateral.

The proof is left to the student.

Note. The case when three sides and the two diagonals are given, can be treated in the same way as case II.

In Exercise XII, § 42, a few more cases have been noticed, and the student is advised to go through that exercise again.

108. Construction of parallelograms.

In the case of the general quadrilateral, we have seen that **five independent data** are required for drawing a quadrilateral in a unique manner. But when we come to the question of construction of parallelograms, **three independent data** are sufficient to fix the construction. The student will have noticed this truth while going through Exercise XI and Exercise XXX. The reason for this is that in a parallelogram, opposite sides are parallel, while there is no such restriction in the case of the general quadrilateral.

We shall notice some particular cases.

(i) **Two adjacent sides and the included angle given.** For such a case see Ex. I, Exercise XXX p. 153.

(ii) **Two adjacent sides and a diagonal given.**

Let x, y be the lengths of the sides, and z that of the diagonal. Construct a triangle **ABC** with its sides equal to x, y, z respectively as shown in Fig. 247. Draw **AD, CD** || to **BC, BA** respectively. **ABCD** is the required parallelogram.

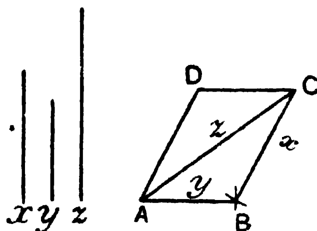


Fig. 247

(iii) Construction of a square on a given line AB.

This is a particular case of (1). Draw **AX** ⊥ to **AB**, cut off **AC** = **AB**.

Draw **CD** ∥ to **AB** and **BD** ∥ to **AC**.

Then **ABCD** is the required square.

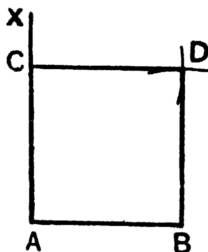


Fig. 248

The proof is left to the student. Note that a square can be constructed uniquely when a side is known.

Note. In the case of parallelograms it is often found convenient to construct from the data the triangle contained by two sides and a diagonal (e. g., the △ **ABC**, Fig. 247). Once this triangle is obtained, the complete parallelogram can be constructed at once.

Ex. 1. Repeat Exercise XXX (Exs. 5—13.)

Ex. 2. Construct a square, the length of a diagonal being given.

Ex. 3. Construct a rectangle the lengths of a diagonal and a side being given.

Ex. 4. Construct a parallelogram, a diagonal being given in position, and the sides given in direction.

Ex 5. Construct a parallelogram **ABCD**, given the side **AB** in position, the length of the diagonal **AC**, and the direction of **AD.**

CHAPTER VIII.

AREAS.

109. Area and its measurement.

The extent of a surface is called its **area.** We speak of the area of a floor or a wall, the area of a plot of land or a field, the area of a sheet of cloth or paper or metal, and so on. When we say that one plot of land is bigger than another we mean that one is greater in area than the other.

In Geometry, a pair of congruent figures are obviously equal in area. Two rectangles of the same breadth and length are equal in area. If two triangles have the three sides of the one equal to the three sides of the other, they are congruent, and therefore equal in area.

But two figures may be equal in area without being congruent.[1]

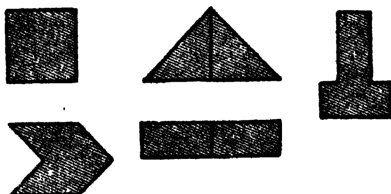


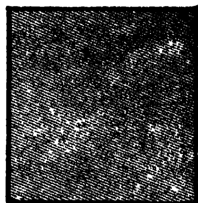
Fig. 249

1. If two figures are equal in area, they are often said to be **equivalent.**

By cutting up the square (Fig. 249) into parts in various ways, and by piecing the parts together in all sorts of ways we may get other figures, all equal in area to the square, though not congruent with it.

110. **Units of area.**

To express the magnitude of an area we must choose a unit of area. The various units of area generally in use are the **square-inch** (the area of a square whose side is 1 inch), the **square-foot** (the area of a square whose side is 1 foot), the **square-yard**, the **square-mile**, and so on. In the metric system the units of area are the **square-centimetre**, the **square-metre** and so on.



1 square inch.
Fig. 250

The following abbreviations should be noted :—

1 sq. in., or 1 in.^{2} means 1 square inch.

1 sq. cm., or 1 cm.^{2} means 1 square centimetre.

1 m.^{2} means 1 square metre,

and similarly for other units.

The units of area to be chosen in any particular case depends on the magnitude of the area to be measured. A large tract of land is measured in sq. miles or acres, a building plot in square yards, the floor of a room in square feet, the size of a photograph in square inches, and so on.

Each of the units of area described above corresponds to a certain linear unit; thus *a square inch* corresponds to the linear unit, *an inch.*, a sq. centimetre to a centimetre, and so on. It is for this reason that

units of area are often called **square units.** Any length might be adopted as the unit of length, the corresponding square-unit would then be the area of a square whose side is this unit of length (Fig. 251).


Fig. 251

111. **The area of a rectangle.** In the rectangle ABCD the side AB = 6 units of length, and BC = 8 units of length. A small division such as DE is the unit of length here, the corresponding square unit is the shaded portion EF. If you count the number of squares in the rectangle, you find the number is 48.

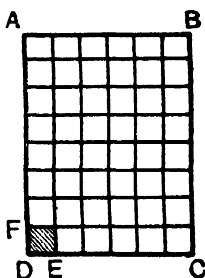


Fig. 252

Thus the area of ABCD is 48 square units. You can get this number more conveniently by counting the number of squares in one row, and multiplying this number by the number of rows, or by counting the number of squares in one column and multiplying this number by the number of columns.

You thus find that the area of ABCD = (6 × 8) square units.

The above result gives us the following rule for finding the area of a rectangle.

If the unit of length is contained a whole number of times in the length as well as in the breadth of a rectangle the number of square units in the area of the

rectangle is obtained by multiplying together the numbers of units (of length) in the length and the breadth of the rectangle.

To establish the rule *when the length and the breadth do not contain the unit of length a whole number of times.*

Take a piece of squared paper (Fig. 253) ruled in squares, one inch each way. There are also finer lines at distances of $\frac{1}{10}$ inch. The paper shows larger squares as well as smaller squares. The larger squares

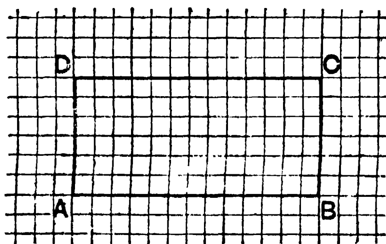


Fig. 253

are all inch-squares and therefore of area 1 sq. inch, and the smaller squares are $\frac{1}{10}$ inch each way. If you look into the paper you will find that each large square (area = 1 sq. in.) contains exactly 100 smaller squares. Thus the area of a small square ($\frac{1}{10}$ in. by $\frac{1}{10}$ in.) = $\frac{1}{100}$ of a sq. inch.

Let us consider the rectangle **ABCD** in which

AB = 0·6 in. i. e. 6 small divisions,

and **AC** = 1·3 in. i. e. 13 small divisions.

area **ABCD** = 6 × 13 small squares.

But 1 small square = $\frac{1}{100}$ sq. inch.

$$\therefore \text{ area ABCD} = \frac{13 \times 6}{100} \text{ sq. in.}$$
$$= (1{\cdot}3 \times 0{\cdot}6) \text{ sq. in.}$$

Thus the number of square inches in the area of the rectangle ABCD is obtained by multiplying the numbers (in this case *fractional*) of inches in the length and the breadth of the rectangle.

Similarly if the sides of a rectangle were for example 1·35 inches and 0·6 inches we could by subdividing the small $\frac{1}{10}$th inch square into 100 smaller squares each with sides of $\frac{1}{100}$ inch, see that the sides of the rectangle were 135 and 60 of these smaller units (namely $\frac{1}{100}$ inch), and the area was 135×60 of the smaller square units ($\frac{1}{100} \times \frac{1}{100}$) or $\frac{1}{10000}$ of a square inch, so that the area of a rectangle is $\frac{135 \times 60}{100 \times 100}$ i.e., $(1{\cdot}35 \times 0{\cdot}6)$ sq. inches.

This area can as before be obtained by multiplying 1·35 by 0·6, and so forth.

We may now state the general rule for obtaining the area of any rectangle :—

Multiply together the numbers (whole or fractional) of units in the length and the breadth of the rectangle, the product gives the number of corresponding square units in the area of the rectangle.

This may be stated in a more abbreviated form

Area = Length × Breadth.

Note. As 1 foot = 12 inches,

1 sq. foot = (12 × 12) sq. inches. Similarly 1 sq. yard = (3 × 3) sq. feet.

1 sq. mile = (1760 × 1760) sq. yards.

1 sq. metre = (100 × 100) sq. centimetres.

Generally, if a certain unit of length is p times as long as another, the square unit corresponding to the former unit of length is $(p \times p)$ i.e., p^2 times as large as that which corresponds to the latter unit of length.

EXERCISE XLIII.

(a)

1. Find the areas of the rectangles of the following dimensions.

(i) 1·2 in. by 3·4 in. (in sq. inches);

(ii) 25 tenths of an inch by 34 tenths of an inch [(a) in *hundredths* of a square inch (b) in square inches];

(iii) 2·3 cm. by 4·5 cm. [(a) in sq. centimetres, (b) in sq. millimetres];

(iv) $2\frac{2}{3}$ in. by $3\frac{1}{4}$ in. [in sq. inches.];

(v) 1·5 ft. by 2 ft. [(a) in sq. feet, (b) in sq. inches.];

(vi) 2·6 ft. by 20 in. [(a) in sq. feet, (b) in sq. inches];

(vii) a in. by b in. [(a) in sq. inches, (b) in sq. feet];

(viii) p cm by q cm. [(a) in sq. centimetres, (b) in sq. metres].

2. Given the area and one of the dimensions, to find the other dimension of the rectangle, in the following cases:

(i) area = 12·6 sq. in., one dimension = 0·9 in

(ii) area = 10 sq. ft., one dimension = 5 in.

(iii) area = 1 acre (4840 sq. yd.), one dimension = 220 yd

(iv) area = 550 sq. cm., one dimension = 1 metre.

(v) area = 1 sq. ft., one dimension = $\frac{1}{16}$ of an inch;

(vi) area = S sq. ft., one side = a inches.

3. What length of paper, a yard in breadth, will be required to cover the four walls of a room whose length = 25 feet, breadth = 18 ft. and height = 14 ft., allowing 200 square feet for doors and windows.

4. How many tiles each 1 ft. by 6 in. are required to cover a roof, 16 ft. by 12 ft.

(b)

5. Draw on inch squared paper a rectangle whose length = 2·3 inches and breadth = 1·4 in Calculate the area and verify by counting the number of small squares.

6. Draw on inch squared paper a rectangle whose area = 3·5 sq. in. and whose breadth = 1·4 in.

7. Draw on inch squared paper a square whose area is 6·25 sq. in.

8. Find the area (in sq. inches) of the figures drawn on inch squared papers (Fig. 254).

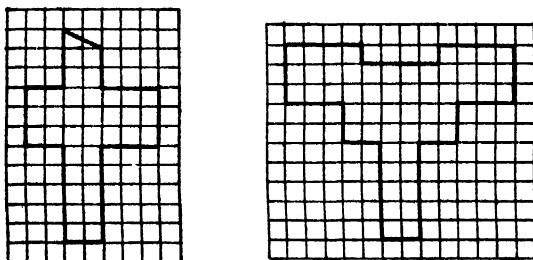


Fig. 254

9. Draw a figure to show that if the side of one square is 5 times the side of another square, the area of the former square is 25 times the area of the latter.

10. Draw on squared paper a rectangle which shall represent the area of a room 12 ft. by 16 ft. on a scale of $\frac{1}{10}$ inch to 1 foot. What is the area of the room? What is the area of the rectangle drawn on squared paper? What actual area does 1 sq. inch on the plan represent?

11. A map is drawn to a scale of an inch to 100 miles. What actual area is represented by 1 sq. inch on the map?

12. A rectangular field is 30 chains long and 24 chains broad. Draw it on inch squared paper on a scale of 1 inch to 10 chains. What area is represented by the diagram?

13. In a scale diagram the sides of a rectangle whose area is an acre are 4·4 in. and 2·75 in. respectively. Find the scale.

14. Find by measurement the areas of the figures given in Fig. 255.

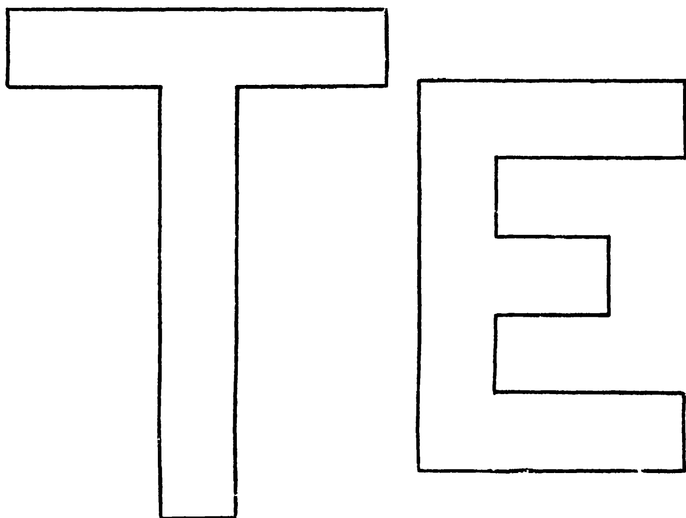


Fig. 255

112. The area of a right-angled triangle.

A glance at the figure on the margin shows how the right-angled triangle **ABC** is just half of the rectangle **ABCD.**

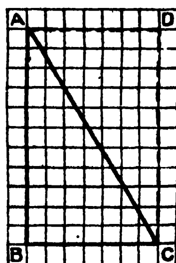


Fig. 256

(This result is proved in Theor. 20 where it is shown that a diagonal bisects a parallelogram).

It follows that the area of the rt. angled △**ABC** is half that of the rect.[1] **ABCD.**

1. The abbreviation '*rect.*' will often be used to mean '*rectangle.*'

If **AB** $= p$ units of length,

and **BC** $= q$ units of length.

Then the area of **ABCD** $= p \times q$ square units (§ 111)

∴ the area of the △**ABC** $= \frac{1}{2}\ pq$ square units. We may state,

The area of a right-angled triangle = half the product of the sides containing the rt. angle.

This statement is to be understood *in the sense that the number of square units in the area of a right-angled triangle is equal to half the product of the numbers of units of length in the sides containing the right angle.*

Ex. Calculate the areas of the right-angled triangles, in which the sides containing the right angle are (i) 3 in., 4 in., (ii) 2·3 in., 3·5 in., (iii) 6·7 cm., 8·5 cm, (iv) 4·25 in., 3·4 in.

113. The area of any triangle ABC.

From **A** draw **AD** prep. to **BC**. Through **A** draw a parallel to **BC**, and through **B**, **E** draw ||s to **AD**. You have thus three rectangles, **ADBE**, **ADCF** and **BCFE**.

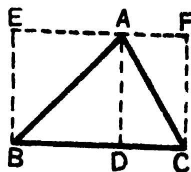


Fig. 257

△**ABD** = $\frac{1}{2}$ rect.**ADBE**.*

* The sign '=' here means '*equality in area*'.

Thus '△**ABD** = $\frac{1}{2}$ rect. **ADBE**' means that the area of the △**ABD** is half the area of the rect. **ADBE**.

$\triangle$**ACD**$=\frac{1}{2}$ rect. **ADCF.**

$\therefore$ the whole $\triangle$ **ABC**$=\frac{1}{2}$ the whole rect. **BCFE.**

$\therefore$ the area of $\triangle$**ABC**$=\frac{1}{2}$ **BC.BE.** (§ 111)

$=\frac{1}{2}$ **BC.AD***

Thus the area of any triangle = half the product of a side and the perpendicular drawn to this side from the opposite vertex.

If the ∠**B** or the ∠**C** of the triangle **ABC** happens to be obtuse as in Fig. 258, the perpendicular **AD** will fall outside the $\triangle$**ABC**. In this case also the above result, viz., the area of $\triangle$**ABC**$=\frac{1}{2}$ **BC. AD**, will hold. For with the same construction as before, we have,

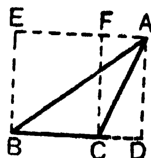


Fig. 258

$\triangle$**ABD**$=\frac{1}{2}$ rect. **ADBE,**

$\triangle$**ACD**$=\frac{1}{2}$ rect. **ADCF** ,

$\therefore$ $\triangle$**ABD**$-\triangle$**ACD**$=\frac{1}{2}$ [rect. **ADBE.** − rect. **ADCF.**]

or $\triangle$**ABC** $=\frac{1}{2}$ rect. **BCFE.**

$=\frac{1}{2}$ **BC.BE.**

$=\frac{1}{2}$ **BC.AD.**

In most cases in books, a triangle is drawn with one of its sides horizontal and at the bottom of the figure. When this is so, the horizontal side can be called the **base,** and the perpendicular drawn to it from the opposite corner, the **altitude** or height of the triangle.

---

* If **BC**$=p$ inches and **BE**$=q$ inches, the product **BC**$\times$**BE** may be written, p in.$\times q$ in., and is to be understood as meaning pq in.^{2} or pq square inches.

(In Fig. 257, if BC were taken as base, then AD would be the altitude). In such cases we may state **that the area of a triangle is half the product of the base and the altitude.**

114. The area of a parallelogram.

Let ABCD be any parallelogram. Through B, C draw perpendiculars to BC meeting the opposite side AD in E and F respectively. Then BCFE is a rectangle. Join AC.

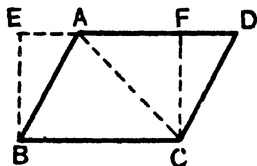


Fig 259

Now rect. BCFE = 2△ABC (§113.)

Also, parallelogram ABCD = 2△ABC, for the diagonal AC bisects the parallelogram (*Theor.* 20).

∴ parallelogram ABCD = rect. BCFE.

∴ area ABCD = BC. BE. or BC. CF.

If the △ CFD (Fig 259) were cut out from the parallelogram **ABCD**, and placed in the position **BEA**, we would get the rectangle **BEFC** * This also shows how the parallelogram **ABCD** is equal in area to the rect. **BEFC**.

Thus the area of a parallelogram ABCD is the product of the side BC and the perpendicular distance between this side and the opposite (parallel) side

Note As in the case of a triangle a parallelogram is generally drawn with a pair of opposite sides horizontal. In this case the bottom side can be called the **base**, and the perpendicular distance between the base and the opposite (parallel) side, the **altitude** or **height**. In (Fig 259), if **BC** were taken as base, then **CF** or **BE** would be the altitude In such a case we may say that *the area of a parallelogram is the product of the base and the altitude.*

---

* It is left to the student to prove that the △s **CFD** and **BEA** are congruent

115. Areas of rectilinear figures.

Any quadrilateral or rectilinear figure may be divided into a number of rectangles and right-angled triangles as shown in Fig. 260, and when this is done, the area of the figure may be calculated by the rules given in §§ 111, 112.

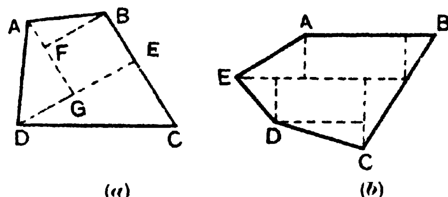


Fig. 260

The quadrilateral **ABCD** [Fig. 260 (*a*)] is divided into a rectangle **EBFG** and three right-angled $\triangle^s$ namely, **ABF, AGD,** and **DEC.** Thus,

area **ABCD** = **EB. BF** + $\frac{1}{2}$ **AF. FB** + $\frac{1}{2}$ **AG. GD** + $\frac{1}{2}$ **DE. EC.**

By measuring **EB, BF, AF, FB,** etc. in inches, and decimals of an inch, you may at once obtain an approximate value of the area of the quadrilateral in square inches.

Fig. 260 (*b*) shows the division of a pentagon into rectangles and right-angled triangles.

Ex. Make three paper quadrilaterals, and cut each of them into rectangles and right-angled triangles, and recover the original quadrilateral in each case by fitting the pieces.

The area of a quadrilateral traced on squared paper.

ABCD is the quadrilateral (Fig. 261). We have a rectangle **LMNP** passing through the vertices and surrounding the quadrilateral as shown in the figure.

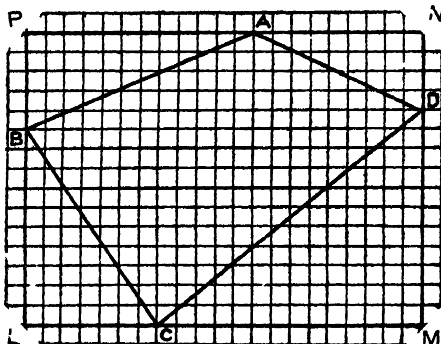


Fig. 261

Then **ABCD** = **LMNP** − **APB** − **BLC** − **CMD** − **DNA**.

$= (21 \times 15 - \frac{1}{2} \times 5 \times 12 - \frac{1}{2} \times 7 \times 10 - \frac{1}{2} \times 14 \times 11 - \frac{1}{2} \times 4 \times 9)$ small squares. $= (315 - 30 - 35 - 77 - 18)$ small squares $= 155$ small squares $= 1{\cdot}55$ sq. in.

The area of a rectilinear figure may also be obtained by dividing it into a number of triangles by joining one vertex to each of the other vertices as shown in Fig. 262.

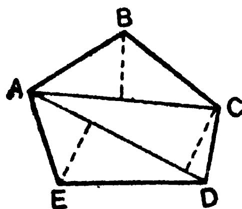


Fig. 262

The area of each of these triangles may be obtained by the rule in § 113, and the sum of the areas of the △s gives the area of the figure. This method is known as **triangulation.**

Let us apply the method to the case of a quadrilateral ABCD. Join AC and draw BE, DF perpendicular to AC.

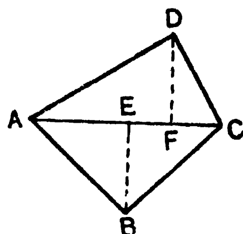


Fig. 263

ABCD = ABC + ADC.
$= \frac{1}{2}$ AC. EB + $\frac{1}{2}$ AC. FD.

If AC = a inches, EB = p inches, FD = q inches,

area ABCD = $\frac{1}{2}$ a p sq. in. + $\frac{1}{2}$ a q sq. in.

$= \frac{1}{2}$ a $(p+q)$ sq. in.

The ⊥s EB and FD are called offsets (with respect to the diagonal AC); and we may state that the area of a quadrilateral = **half the product of a diagonal and the sum of the corresponding offsets.**

The area of a trapezium.

ABCD is a trapezium (Fig. 264) AB, CD being the parallel sides. Draw AE, CF ⊥ to AB. ABCD = ACB + ACD
$= \frac{1}{2}$ AB. CF + $\frac{1}{2}$ CD. AE.

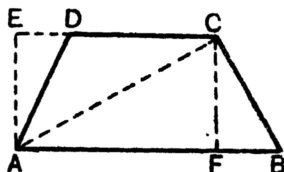


Fig. 264

If AB = a in., CD = b in., and AE = CF = p in., we have area ABCD = $\frac{1}{2}$ a p sq. in. + $\frac{1}{2}$ b p sq. in. = $\frac{1}{2}$ p $(a+b)$ sq. in.

Thus, the area of a trapezium = half the sum of the parallel sides into the perpendicular distance between them.

EXERCISE XLIV.

In Exs. 1—11, construct the figures as indicated and find the area of each figure, by drawing such lines and taking such measurements as may be necessary —

1. A rectangle 2 in. by 3 in.

2. A right-angled Δ, (i) of sides 2 in. and 3·5 in. (ii) of sides 4·5 cm. and 6·2 cm.

3. A triangle **ABC**; **AB**$=2$ in., **BC**$=2{\cdot}5$ in., **CA**$=3$ in.

4. A triangle **ABC**; **AB**$=5$ cm., **BC**$=7$ cm., $\angle$**A**$=48^\circ$.

5. A triangle **ABC**; **BC**$=3$ in., $\angle$**B**$=50^\circ$, $\angle$**C**$=60^\circ$

6. A triangle; base$=5$ in., altitude$=2{\cdot}6$ in.

7. A parallelogram **ABCD**: **AB**$=5$ cm., **AD**$=6$ cm., and $\angle$**A**$=45^\circ$.

8. A parallelogram; adjacent sides 3 in., and 2·7 in., included angle 30°.

9. A rhombus; diagonals 6 cm., and 8 cm.

10. A trapezium; parallel sides 2 in., 3 in., and the perpendicular distance between them 1·5 in.

11. A quadrilateral; a diagonal$=7$ cm., and the perpendicular distances of the opposite vertices from it$=3$ cm. and 2 cm. respectively.

12. Make copies of the figures given below, and obtain their areas by triangulation.

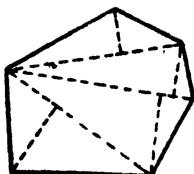


Fig. 265

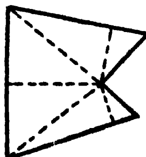


Fig. 266

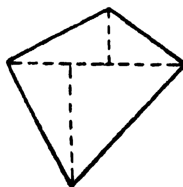


Fig. 267

13. Draw figures whose shapes and dimensions are indicated by the rough plans given below and calculate the areas.

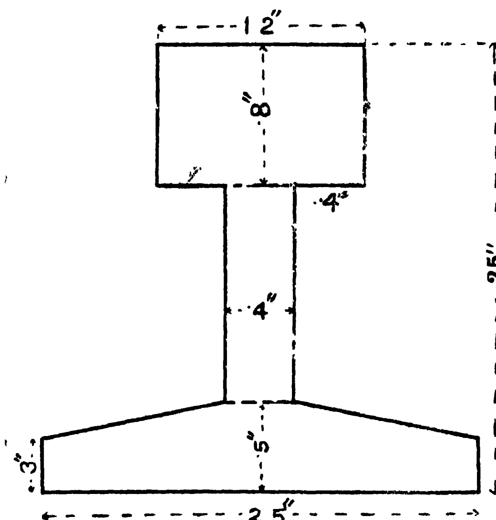


Fig. 268

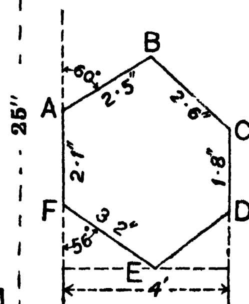


Fig. 269

14. Draw on a scale of 1 inch to 10 feet a correct plan of the figure whose shape and dimensions are indicated by the rough plan given in Fig 270, find the area of the plan you draw, and deduce from it the area of the true figure.

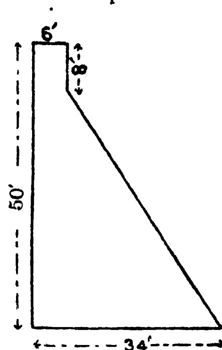


Fig. 270

116. The surveyor's method of finding the area of a field. The Field-Book.

The field is regarded as a rectilinear figure, prominent points in the field being so chosen that they determine the vertices of a polygon which approxi-

mately represents the boundary of the field as shown in Fig. 271, ABCDE being the field.

The surveyor chooses a suitable line AC for the **base-line.** He proceeds along AC and marks the points P, Q, R, from which the points E, B, D are seen in

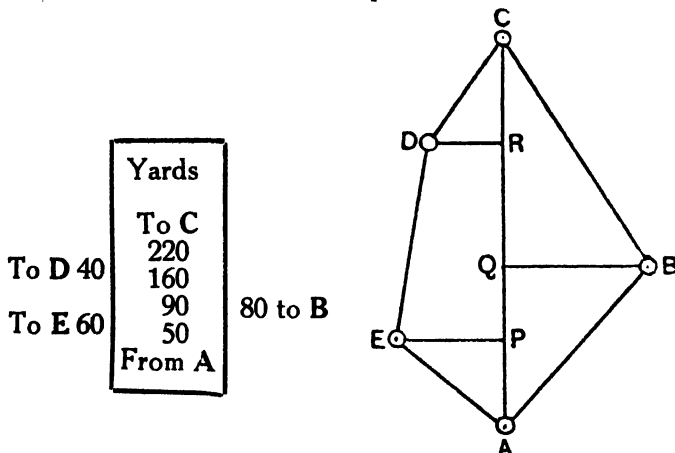


Fig. 271

directions at right angles to AC. The distances AP, AQ, AR, AC as well as the **offsets** PE, QB, RD are measured with a chain, and the measurements are recorded in what is called the Field-Book, as shown above. The distances on the base-line are recorded in the central column, and the measures of the offsets are recorded on the right or on the left of this column according as they are to the right or the left of the base line.

The base-line and the offsets divide the figure into right-angled triangles and trapeziums, and the area of the field may be calculated as follows.

ABCDE = △**AEP** + **trap**^{m}. **EPRD** + △**CDR** + △**AQB** + △**QBC**.

***Calculation*:**

△AEP = ½ AP. EP	= ½ × 50 × 60	= 1500
trap^{m}. EPRD = ½ RP. (EP + DR).	= ½ × 110 × 100	= 5500
△CDR = ½ CR. DR	= ½ × 60 × 40	= 1200
△AQB = ½ AQ. QB	= ½ × 90 × 80	= 3600
△QBC = ½ QC. QB	= ½ × 130 × 80	= 5200

Adding together, the area of the field = 17000 sq. yds.

$$= \frac{17000}{4840} \text{ acres.}$$

$$= 3{\cdot}5 \text{ acres nearly.}$$

Note. Gunter's **chain** (used by surveyors) is 22 yards long and is divided into 100 **links**, so that a link = 0·22 yard, 1 **acre** = 4840 sq. yards = 10 **sq. chains** = 100000 **sq. links.***

EXERCISE XLV.

1. Calculate the area of the field **ABCDEF** from the rough plan, Fig. 272, and the Field-Book entry, given above.

	Yards.	
	To D	
	400	
	360	60
65	315	
	260	
	290	170
	From A	

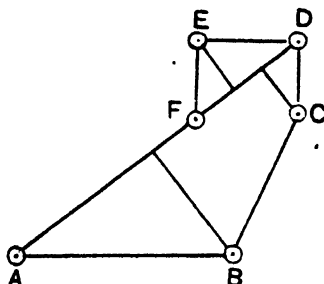


Fig. 272

* For general use a *chain* of 100 feet divided into 100 links of 1 foot each is the most convenient. But when the sole object is to obtain the acreage, Gunter's chain is specially adopted. (*Roorkee Manual—Surveying*).

2. Draw (*free-hand*) rough plans of the fields from the following Field-Book entries.

(i)

	Metres	
	To C	
	250	
	150	20
30	120	
	100	
10	60	
	From **A**	go north

(ii)

	Links.	
	To **C**	
	990	
	725	
	460	150
85	220	
From	**A**	go east.

(iii)

	Links.	
	To **E**	
	2750	
	1600	500
600	950	
	600	400
	400	200
From	**A**	go **S. W.**

Calculate the area in sq. metres in case (i) and in sq. links in cases (ii) and (iii). Express the areas in acres also, in the last two cases.

CHAPTER IX

THEOREMS ON AREAS AND ALLIED PROBLEMS.

THEOREM 22 [Euclid 1, 35]

117. **Parallelograms on the same base and between the same parallels (or of the same altitude)[1] are equal in area**

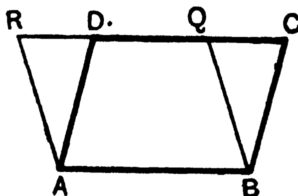


Fig. 273

ABCD and **ABQR** are two parallelograms on the same base **AB** and between the same parallels **AB, RC.**

To prove that ABCD, ABQR are equal in area.

Proof. In the △s RAD, QBC

∠ ARD = corresp. ∠ BQC, [∵ AR,BQ are ∥],
∠ ADR = corresp. ∠ BCQ, [∵ AD,BC are ∥],
AD = BC (opp. sides of ∥m ABCD).

∴ the △s are congruent.

Now if △ARD is taken away from the whole figure ABCR, the ∥m ABCD is left ; and if △ BQC is taken away from the same figure ABCR, the ∥m ABQR is left.

∴ ∥ms ABCD and ABQR are equal.

(1) For the meaning of the term '*altitude*' see § 114

(2) We shall often use the abbreviation '∥m for *parallelogram.*

Note. The ||ms **ABCD, ABQR** might be situated, as shown in Figs. 274, 275. The same proof covers these cases also.

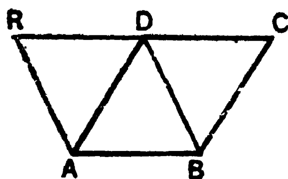


Fig. 274

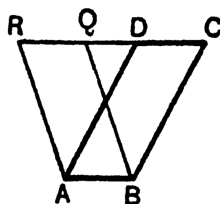


Fig. 275

Corollary 1.—Parallelograms on **equal bases and of the same altitude** are equal in area. (For we can place them so as to be on the same base and between the same parallels.)

Corollary 2.—A parallelogram is equal in area to the rectangle on the same base and of the same altitude (See § 114). This gives us the following rule for the area of a parallelogram.

area = base × altitude.

The area of any parallelogram may be obtained by multiplying any side by the perpendicular distance between this side and the opposite (parallel) side. Thus the area of a parallelogram ABCD may be obtained in two ways, (i) as the product AB. EF, or (ii) as the product BC. GH. These two products must be equal, since each of them gives the area of the parallelogram. But in practice they are not found to be exactly equal owing to imperfect measurements. The best course is to take the average of the two results.

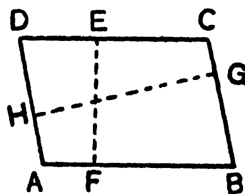


Fig. 276

EXERCISE XLVI.

1. Construct a parallelogram with sides 2″ and 1·5″ and the included angle 40°. Measure the perpendicular distance between each pair of opposite sides and make two independent calculations of the area, and find their average.

2. Draw a rectangle of base 3 inches and altitude 1·8 in. On the same base construct a parallelogram of the same area as the rectangle, and with one of its angles 50°.

3 In Ex. 2, on the same base as the rectangle, construct a rhombus equal in area to the rectangle.

4. One side of a parallelogram is 8 cm., and its area is 40 sq. cm. Calculate its altitude with respect to this side as base. Construct the parallelogram if the other side is 7 cm.

5. A rhombus of side 2 inches has its area 3 sq. inches. Calculate the perpendicular distance between two opposite sides, and construct the rhombus.

6. A rectangle one of whose sides is 2 inches has the same area as a 3-inches square. Calculate the other side of the rectangle and construct it.

7. Transform a parallelogram of sides 2 in. and 3 in. and acute angle 40°, into a rhombus of equal area, of side 3 inches Measure the acute angle of the rhombus

8. Two parallelograms have each their adjacent sides 2 in. and 3 in, the acute angle of the one is 30°. that of the other, 40°; construct the parallelograms and compare their areas.

9. **If two parallelograms have the same altitude, their areas are in the same ratio as the bases.**

10 **If two parallelograms have equal bases, their areas are in the same ratio as their altitudes.**

11. Two adjacent sides of a plot of land in the form of a parallelogram are 100 ft. and 150 ft. and the included angle is 50°. Draw a plan on a scale of 1 inch to 50 feet, and obtain from it the area of the plot in square feet.

THEOREM 23. [Euclid 1, 37.]

118. **Triangles on the same base and between the same parallels (or of the same altitude) are equal in area.**

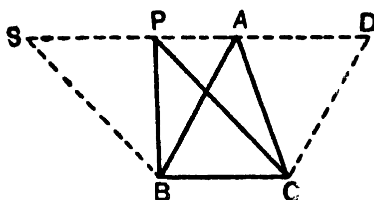


Fig. 277

ABC, PBC are two △ˢ on the same base **BC** and between the same parallels **BC, DS.**

To prove that the △ˢ are equal in area.

Through **C** draw **CD** ∥ to **BA**, through **B** draw **BS** ∥ to **CP**; let **CD, BS** meet **AP** (produced if necessary), in **D** and **S** respectively.

Proof. △**ABC** = ½ ∥ᵐ **ABCD** [∵ the diagonal **AC** bisects the ∥ᵐ **ABCD**, Theor 20.];

Similarly △**PBC** = ½ ∥ᵐ **BCPS** „

But ∥ᵐˢ **ABCD, BCPS** are equal in area being on the same base **BC** and between the same parallels.

Hence △ˢ **ABC, PBC** are equal in area.

Corollary 1.—Triangles on equal bases and of the same altitude are equal in area.

(For we can place them so as to be on the same base and between the same parallels).

Corollary 2.—If a triangle and a parallelogram be on the same base and between the same parallels the area of the triangle is half that of the parallelogram.

$\triangle$**ABC**, and $\|^m$ **BCPS** stand on the same base **BC**, and between the same parallels **BC, SA** (Fig. 277).

Join **PB**. Now because **BC** and **SA** are parallel,

$\therefore$ $\triangle$**ABC** = $\triangle$**PBC** (*Theor.* 23).

But $\triangle$**PBC** = $\frac{1}{2}$ $\|^m$ **BCPS**,

$\therefore$ $\triangle$**ABC** = $\frac{1}{2}$ $\|^m$ **BCPS.**

Corollary 3.—The area of a triangle is half the area of the rectangle on the same base and of the same altitude. This gives us the following rule for calculating the area of a triangle:

area = $\frac{1}{2}$ base × altitude (See § 113).

It should be observed that the area of a triangle may be obtained by multiplying any side by the perpendicular drawn to it from the opposite vertex and taking half this product. If **AD, BE, CF** are the perpendiculars from **A, B, C** to the opposite sides **BC, CA, AB** (Fig. 278), then the products $\frac{1}{2}$ **BC . AD**, $\frac{1}{2}$ **CA . BE**, and $\frac{1}{2}$ **AB . CF** will each give the area of the $\triangle$ **ABC**, hence they must be equal. In practice, however owing to imperfect measurements, they may not be found to be exactly equal. The best course is to take the average of the three results.

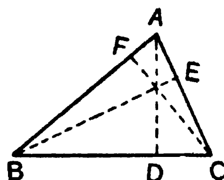


Fig. 278

EXERCISE XLVII.

1. Construct triangles **ABC** to the following measurements. Measure in each case, the perpendiculars **AD, BE, CF** (see Fig. 278) and make three independent calculations of the area, and find the average of the three results.

(i)	**AB**=2 in.,	**BC**=2·5 in.,	**CA**=1·8 in.
(ii)	**AB**=6 cm.,	**BC**=8 cm.,	**CA**=6 cm.
(iii)	**AB**=2 in.,	**BC**=2 in.,	**CA**=2 in.
(iv)	**AB**=3 in.,	**BC**=3 in.,	**CA**=3 in
(v)	**AB**=6 cm.,	**AC**=8 cm.,	∠**A**=30°.
(vi)	**BC**=7 cm.,	∠**B**=60°,	∠**C**=40°.

2. Find the ratio of the areas of the two equilateral triangles in (iii) and (iv) of Ex. 1.

3. Find the area of the △**ABC** in Fig. 107.

4. Find the area of the △**XYZ** in Fig. 108.

5. Find the areas of the two △s, ABC_1 and ABC_2 in Fig. 109.

6. The sides of a triangular plot of land measure 200 ft., 250 ft., and 300 ft. Draw a plan on a scale of 1 inch to 100 feet, and obtain from it the area of the plot in square feet.

7. A triangle **ABC** has its area=3 sq. in, and the sides **AB, AC**=2 in. and 3·5 in. respectively. Construct the triangle and measure **BC**. [Calculate an altitude]. How many solutions are possible ?

8. A triangle **ABC** has its area=3 sq. in., side **AB**=2 in., and ∠**A**=45°. Construct the triangle and measure **AC, CB**

EXERCISE XLVIII.

1. *Show that a median of a triangle bisects it.*

2. If two triangles have two sides of the one equal to two sides of the other, and the included angles supplementary, the triangles are equal in area.

3. **ABCD** is a parallelogram, and **O** is any point on **AB**, show that the △ **COD**=½ ||^{m} **ABCD.**

4. **BE**, **CF** are two medians of a $\triangle$**ABC**, intersecting in the point **G**. Show that (i) $\triangle$**BEF** $= \frac{1}{4}$ $\triangle$**ABC**, (ii) $\triangle$ **BEF** $=$ $\triangle$**CFE**, and (iii) $\triangle$**BGF** $=$ $\triangle$**CGE**.

5. If **X**, **Y** are mid-points of the sides **PQ**, **PS** of a parallelogram **PQRS**, show that $\triangle$**PXY** $= \frac{1}{8}$ of the $\|^{m}$ **PQRS**.

6. **PQ** is any straight line, and **O** any point outside **PQ**. **R** is a point on **PQ** such that **PR** $= \frac{1}{4}$ **PQ**. Show that $\triangle$**OPR** $=$ $\frac{1}{4}\triangle$**OPQ**.

7. In Ex. 5, if **S** and **T** are two points in **PQ**, such that **ST** $=$ $\frac{3}{5}$ **PQ**, then $\triangle$**SOT** $= \frac{3}{5}$ $\triangle$**POQ**.

8. *If two triangles have the same altitude, their areas are proportional to their bases.*

9. *If two triangles have equal bases, their areas are proportional to their altitudes.*

10. If **D**, **E**, **F** are the mid-points of the sides of a triangle **ABC**, show that $\triangle$ **DEF** $= \frac{1}{4}$ $\triangle$**ABC**.

11. **X** is the mid-point of the side **QR** of a parallelogram **PQRS**. **PX** cuts the diagonal **QS** in **T**. Show that $\triangle$**PTS** $=$ $\frac{1}{3}$ of $\|^{m}$ **PQRS**. [See Ex. 7, p. 228].

12. **ABCD** is a parallelogram and **O** is any point in **CD**, show that $\triangle$**AOC** $=$ $\triangle$**AOB** $-$ $\triangle$**AOD**.

13. **ABCD** is a parallelogram and **O** is any point in **CD** *produced* Show that $\triangle$**AOC** $=$ **AOB** $+$ $\triangle$**AOD**.

14. Show that the area of a rhombus is half the product of its diagonals.

15. If the diagonals of a quadrilateral are at right angles the area of the quadrilateral is half the product of the diagonals.

16. If a, b, c, denote the lengths of the sides **BC**, **CA**, **AB** of a triangle **ABC**, and x, y, z be the distances of any point **O** within the triangle from **BC**, **CA**, **AB** respectively, show that $ax+by+cz$ has the same value wherever the point **O** may be taken within the triangle.

THEOREM 24. [Euclid 1, 40]

119. **If two triangles of equal areas, stand on the same base, or on equal bases in the same straight line, and are on the same side of it, they are between the same parallels**

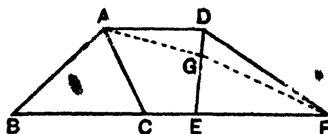


Fig 279

$\triangle$s **ABC, DEF** of equal areas stand on equal bases* **BC, EF** which are in one straight line **BF**, the vertices **A** and **D** being on the same side of **BF**.

To prove that **AD** is parallel to **BF.**

Proof. If **AD** is not parallel to **BF**, suppose **AG** is the line through **A**, parallel to **BF**, and let this line cut **ED** (or **ED** produced) in **G**. Join **FG**.

Since **BC** = **EF**, and **AG** is ‖ to **BF**,

$\therefore$ $\triangle$**ABC** = $\triangle$**GEF**. (*Theor.* 23.)

But $\triangle$**ABC** = $\triangle$**DEF**. (*given*)

$\therefore$ $\triangle$**GEF** = $\triangle$**DEF**.

This is absurd, for of the $\triangle$s **DEF, GEF** one is a part of the other.

Hence AD is parallel to BF.

* We take the case of two $\triangle$s standing on equal bases in a straight line. The case of two $\triangle$s standing on the *same* base, is proved in exactly the same way.

Corollary 1. Triangles of equal areas, standing on the same base or on equal bases, are of the same altitude. [For if the $\triangle^s$ were placed with their bases in the same st. line, and the vertices on the same side of this line, they would be between the same parallels.]

Note—The above result might be deduced from the property indicated in Corollary 3 to Theorem 23, viz., *the area of a* $\triangle = \frac{1}{2}$ *base* $\times$ *altitude.* If we consider two triangles whose bases are a, a' units of length and altitudes p, p' units of length and have areas $\mathbf{A}$, $\mathbf{A}'$ square units respectively, we have

$$\mathbf{A} = \tfrac{1}{2}\, ap$$
$$\mathbf{A}' = \tfrac{1}{2}\, a'p'$$

Now if $\mathbf{A} = \mathbf{A}'$, and $a = a'$ then p must be equal to p'.

Ex. 1. If two triangles of equal areas, are on opposite sides of a common base, the line joining their vertices is bisected by the base.

Ex. 2. If E, F are the mid-points of the sides AC, AB of a $\triangle$ABC, show that EF is parallel to BC.



120. We shall now take up some problems.

PROBLEM 8.

To construct a triangle equal in area to a given quadrilateral.

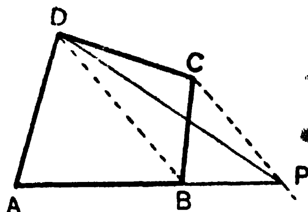


Fig. 280

ABCD is a quadrilateral ;

To construct a triangle equal to it in area.

Construction. Join **BD** ; through **C** draw **CP** parallel to **DB**, meeting **AB** produced in **P**.

Join **DP**.

Then **APD** is the required triangle.

Proof. Since **BD** and **CP** are parallel,

$\triangle$**DBP**$=\triangle$**BCD**. (*Theor.* 23)

To each add $\triangle$ **ABD**,

$\therefore$ $\triangle$ **APD** = quadrilateral **ABCD**.*

Note. In a similar way we may construct a quadrilateral equal in area to a given pentagon, a pentagon equal in area to a given hexagon, and so on ; in other words, we can construct a rectilinear figure equal in area to a given rectilinear figure, and having fewer sides by one than the given figure.

---

* A similar construction holds when the angle **C** of the quadrilateral **ABCD** is reflex (see Fig. 103). The proof is left to the student.

121. To construct a triangle equal in area to a given rectilinear figure, say, the pentagon ABCDE (Fig. 281).

[We shall first construct a quadrilateral equal in area to the given pentagon, and then a triangle equal in area to this quadrilateral, and hence equal in area to the given pentagon].

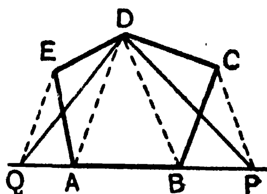


Fig. 281

Join **DB**; through **C** draw a parallel to **DB** meeting **AB** in P. Join **DP**. Then the quadrilateral **APDE** is equal in area to the given figure **ABCDE** (proof as in Problem 8). Now join **DA**; through **E** draw **EQ** || to **DA**, meeting **PA** in **Q**. Join **DQ**. Then △ **DPQ** is equal in area to the quadrilateral **APDE**. Thus △ **DPQ** is equal in area to the given pentagon **ABCDE**.

Note—The above process is general, and a rectilinear figure of any number of sides can by a course of successive reduction be ultimately reduced to a triangle of equal area. We can find the area of this triangle by the rule given in § 113 or in Cor. 3, of § 118, and thus obtain the area of the original rectilinear figure.

EXERCISE XLIX.

1. Draw quadrilaterals as indicated in Exs. 1 to 5, p. 87. Reduce each of them to a triangle of equal area; and measuring a side of this triangle, and the perpendicular drawn to it from the opposite vertex, find an approximate value of the area of the quadrilateral.

2. **PQRS** is a quadrilateral with the ∠**R**, a reflex angle (see Fig. 101). Construct a triangle equal in area to **PQRS**.

3. Draw a pentagon **ABCDE**, such that **AB**=**CD**=**DE**=1·5 in., **BC**=**AD**=1 in., and ∠**A**=∠**B**=120°.

Reduce the pentagon to a triangle of equal area, and measuring a side of this triangle, and the perpendicular drawn to it from the opposite vertex, obtain an approximate value of the area of the pentagon.

4. Construct a regular hexagon of side 1·5 in ; reduce it to a triangle of equal area, and thence obtain the area of the hexagon.

5. **Show that the area of a trapezium=(half the sum of the parallel sides) × (the perpendicular distance between them).** [Through the mid-point (**E**) of **BC** draw **PQ** parallel to **AD** (Fig. 282). Show that trapm **ABCD**=||m **APQD** by proving that △**PEB**=△**QEC**.

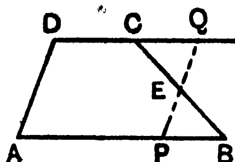


Fig. 282

If you can now show that **AP**=½ (**AB**+**CD**), you will prove the truth of the statement. See p. 259.]

PROBLEM 9.

122. To construct a parallelogram equal in area to a given triangle and having one of its angles equal to a given angle.

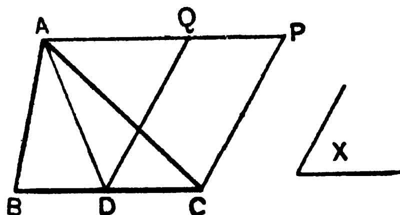


Fig. 283

To construct a $\|^{m}$ equal in area to $\triangle$ **ABC**, and having one of its angles equal to $\angle$**X**.

Construction :—Bisect **BC** at **D**. Through **D** draw **DQ** making $\angle$ **CDQ** = $\angle$ **X**.

Through **C** draw **CP** $\|$ to **DQ** ; through **A** draw a parallel to **BC** cutting **CP**, **DQ** in **P**, **Q**.

Then **DCPQ** is the required parallelogram. Join **AD**.

Proof. Since $\triangle$ **ADC** and $\|^{m}$ **DCPQ** are on the same base **DC** and between the same parallels **DC**, **AP**,

$\therefore$ $\|^{m}$**DCPQ** = 2 $\triangle$ **ACD**. (*Theor.* 23, *Cor.* 2)

Again, since **D** is the mid-point of **BC**,

$\therefore$ $\triangle$ **ABD** = $\triangle$ **ACD** :

$\therefore$ $\triangle$ **ABC** = 2$\triangle$ **ACD** ;

Hence $\|^{m}$ **DCPQ** is equal to $\triangle$ **ABC**, and has its angle **QDC** equal to the given $\angle$**X**.

123. An important property of a parallelogram.

ABCD is a $\|^m$ (Fig. 284). O is any point on one of the diagonals, say BD. Through O, parallels EF, GH are drawn to the sides AB, AD.

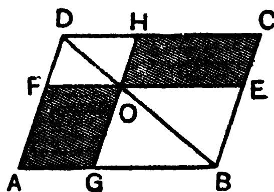


Fig. 284

The whole $\|^m$ ABCD is now divided into four parallelograms, viz. FG, HE, FH, GE.

Now $\triangle$ BAD = $\triangle$ BCD, $\triangle$ OFD = $\triangle$ OHD, $\triangle$ OGB = $\triangle$ OEB. (*Theor.* 20)

Hence $\triangle$ BAD − $\triangle$ OFD − $\triangle$ OGB = $\triangle$ BCD − $\triangle$ OHD − $\triangle$ OEB.

$\therefore$ $\|^m$ FG = $\|^m$HE.*

Add to each the $\|^m$ GE,

$\therefore$ $\|^m$ ABEF = $\|^m$ BCHG.

It may be similarly shown that

$\|^m$ ADHG = $\|^m$ CDFE.

Problem. To construct a parallelogram which shall be equal in area to a given parallelogram, and have one of its sides of a given length.

---

* The $\|$ms **FH, GE** are called **parallelograms about the diagonal BD,** and the other two parallelograms, viz., **EH, FG** are called the **complements** of the parallelograms about the diagonal. It is proved here that the complements of the parallelograms about a diagonal are equal in area.

Let **ABCD** be a $\|^m$ (Fig. 285); to construct a parallelogram equal in area with it, and having a side of given length x.

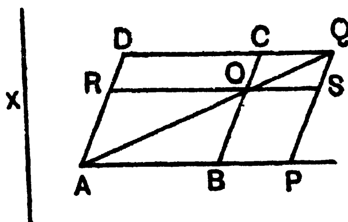


Fig. 285

Cut off from **AB** (or **AB** produced) a length **AP** $= x$.

Through **P** draw **PQ** $\|$ to **BC** cutting **DC** in **Q**.

Join **AQ**, and let **AQ** cut **BC** in **O**. Through **O** draw **RS** $\|$ to **AB** cutting **AD, PQ** in **R, S.**

Then **APSR** is the required parallelogram, for the $\|^m$ **APSR** $= \|^m$ **ABCD** (by the property proved above), and has one of its sides, **AP**, equal to x; moreover its angles are equal to those of the $\|^m$ **ABCD.**

Note 1. Once *one* parallelogram has been obtained, equal in area to a given $\|$m, and having one of its sides of length x, we can (by Theor. 22) construct an infinity of other parallelograms all with one of the sides$=$x, and equal in area to the given $\|$m.

Note 2. With the aid of the above problem combined with Problem 9 (§ 122) the student will be in a position to construct a parallelogram which shall be equal in area to a given triangle, have one of its sides of a given length, and one of its angles equal to a given angle.

EXERCISE L.

1. Draw rectangles equal in area to the triangles constructed in Ex. 1, Exercise XLVII.

2. Draw a rectangle which shall be equal in area to an equilateral triangle of side 2 inches.

3. Construct a rectangle equal in area to a given rectangle **PQRS**, and having one side equal to a given line **AB**.

4. Draw a rectangle on a base 5 cm. equal in area to a rectangle 4 cm. by 6 cm. [You are not to calculate the other side of the required rectangle. Construct as explained in § 123, and then measure the other side, and verify by calculation.]

5. Draw a rectangle (one of the sides to be 1·5 in.) equal in area to a square of side 2 in. [You are not to calculate the other side of the rectangle before construction.]

6. Construct a parallelogram equal in area to a given rectangle **PQRS**, and having one side equal to a given line **AB**, and one angle equal to a given angle **X**.

7. Construct a rhombus of side 4 in., equal in area to a parallelogram of adjacent sides 2 in. and 3 in. and included $\angle 60°$, (without calculating the perpendicular distance between opposite sides of the required rhombus before construction).

8. Construct a rectangle equal in area to a given $\triangle$**ABC**, and having one of its sides of a given length **PQ**.

9. Construct a parallelogram equal in area to a given $\triangle$**ABC**, and having one of its sides equal to a given line **PQ** and one of its angles equal to the $\angle$**B** of the $\triangle$ **ABC**.

PROBLEM 10.

124. **To construct a parallelogram equal in area to a given quadrilateral and having an angle equal to a given angle.**

[By problem 8, you can construct a triangle equal in area to the given quadrilateral, and then by problem 9, you can construct a parallelogram equal in area to this triangle, and having an angle equal to the given angle.]

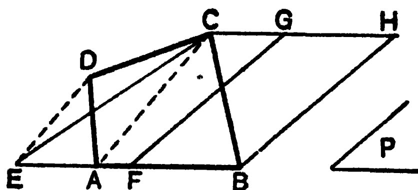


Fig. 286

ABCD is a given quadrilateral and **P** the given angle.

By problem 8, the △ **EBC** is constructed equal in area to **ABCD.**

By problem 9, the ||m **BFGH** is constructed equal in area to △ **EBC**, and having ∠**GFB** = ∠**P.**

Evidently ||m **BFGH** = quad. **ABCD.**

Note. Indeed a parallelogram may always be constructed which shall be equal in area to any given rectilinear figure, and have one of its angles equal to a given angle.

[The given figure is first reduced to a triangle (see §121, note), and then a parallelogram is constructed equal in area to this triangle and having an angle equal to the given angle.]

Ex. 1. Construct a parallelogram equal in area to a given rectilinear figure (say, a pentagon), and having one of its angles

equal to a given angle. [Reduce the given figure to a triangle of equal area, and then use problem 9.]

Ex. 2. Construct a rectangle equal in area to a given rectilinear figure (say, a hexagon), and having one of its sides equal to a given line.

125. Applications.

We shall consider two problems ·

(*a*) **To bisect a triangle ABC by a straight line drawn through a given point O in one of its sides, BC.**

Join **AO** (Fig. 287) and through **D**, the mid-point of **BC**, draw **DP** || to **OA** meeting **AB** in **P**. Join **OP**. Then **OP** bisects △**ABC**. Join **AD**, and prove that △ **BPO** = △ **ABD**. But △ **ABD** = $\frac{1}{2}$ △ **ABC** as **D** is the mid-point of **BC**. Hence △ **BPO** = $\frac{1}{2}$ △ **ABC**. Thus **OP** divides △ **ABC** into two equal parts. (If **O** were the mid-point of **BC**, **OA** would bisect the triangle).

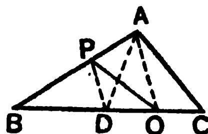


Fig. 287

(*b*) **To trisect a triangle ABC by straight lines drawn through a given point O in one of its sides, BC.**

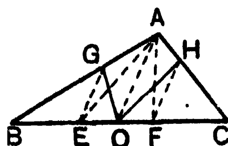


Fig. 288

Join **OA** and trisect **BC** at **E**, **F** (Problem 7 p. 225). Draw **EG**, **FH** || to **OA** meeting **AB**, **AC** respectively in **G**, **H**. Join **OG**, **OH**. Then **OG**, **OH** divide the △ **ABC** into three equal parts.

Proof. By joining **AE**, **AF**, it is easily proved that △ **OBG** = △ **ABE**, and △ **OCH** = △ **ACF**. But △s **ABE**, **ACF** are each $\frac{1}{3}$ △ **ABC**, ∴ △s **OBG**, **OCH** are each a third of the △ **ABC**.

[Consider the case when **O** coincides with **E**, a point of trisection of **BC**.]

Note. If **BE** (Fig. 288) were an *n*th part of **BC**, △ **ABE** would be an *n*th part of the △ **ABC**, ∴ △ **GOB** which is equal to △**ABE** would also be an *n*th part of △**ABC**. Thus the line **OG** drawn through the given point **O** would cut off from the △**ABC**, an *n*th part of it. [Consider the case when **O** lies between **B** and **E**.]

EXERCISE LI.

(Miscellaneous)

(*a*)

1. Show that any straight line drawn through the centre* of a parallelogram bisects the parallelogram.

2. The medians **BE**, **CF** of a $\triangle$**ABC** intersect in **G**, show that $\triangle$**BGC** = quad **AEGF**.

3. A square and a rhombus stand on the same base, show that the area of the square is greater than that of the rhombus.

4. Of all parallelograms constructed with adjacent sides of the same given lengths, the rectangle is of the greatest area.

5. **ABCD** is any quadrilateral. Through **A**, **C**, lines ‖ to **BD** are drawn, and through **B**, **D** lines ‖ to **AC** are drawn. Show that the parallelogram formed by these lines is double the quadrilateral in area.

6. **P**, **Q**, **R**, **S** are the mid-points of the sides **AB**, **BC**, **CD**, **DA** of any quadrilateral **ABCD**. Show that **PQRS** = $\frac{1}{2}$ **ABCD**.

7. **ABCD** is a parallelogram, and **E** is any point on the diagonal **BD**. Show that $\triangle$ **AED** = $\triangle$**DEC**.

8. **ABCD** is any parallelogram, and **E**, **F** are any two points on the diagonal **BD**, show that $\triangle$s **AEF** and **CEF** are equal in area.

9. Show that the line joining the mid-points of the parallel sides of a trapezium bisects the trapezium.

10. **AB**, **CD** are the parallel sides of a trapezium **ABCD**, and **E** is the mid-point of **BC**. Show that $\triangle$**AED** = $\frac{1}{2}$ trap^m **ABCD**.

11. **ABCD** is a trapezium, in which **AB**, **CD** are parallel sides, and **E** is the mid-point of **BC**. **DE** is joined and produced to meet **AB** produced in **L**. Show that $\triangle$ **ADL** is equal in area to the trapezium.

---

* The intersection of the diagonals may be called the **centre** of the parallelogram.

12. **O** is any point within a parallelogram **ABCD.** Show that $\triangle$**AOB**+$\triangle$**COD**=$\triangle$**BOC**+$\triangle$**AOD**=$\frac{1}{2}$ ||m **ABCD.**

13. **ABCD** is any parallelogram and **O** is any point within it. Show that $\triangle$**BOD** is equal to the difference of the $\triangle$s **AOB** and **BOC.**

14. From any point within an equilateral triangle perpendiculars are drawn to the sides; show that the sum of these perpendiculars is the same wherever the point is taken.

15. **ABC** is a triangle, and **BP** and **CQ** are two parallel lines drawn through **B** and **C**. **BP** cuts **AC** produced in **P**, and **CQ** cuts **AB** in **Q**. If $\triangle$**BPC**=$\triangle$**ABC**, show that **Q** is the mid-point of **AB**.

(*b*)

16. Divide a parallelogram into 5 equal parallelograms.

17. **ABC** is a triangle, and **P** and **Q** are points on **AB**, **AC**, respectively such that **AB**=m. **AP**, and **AC**=n **AQ**; show that $\triangle$**ABC**=mn $\triangle$**PAQ**.

18. Show how to divide a triangle into any number of equal parts by straight lines drawn through one of the vertices.

19. Show how to bisect a parallelogram by a line drawn through any given point. (See Ex. 1.)

20. Bisect a parallelogram by a line drawn perpendicular to one of its sides. (See Ex. 1.)

21. Divide a triangle into two parts in the ratio 3 : 4 by a straight line drawn through a vertex.

22. **ABC** is a triangle, and **P** is any point in **BC**. Show how to draw a straight line through **P** to meet **BA** produced in **Q** so that $\triangle$**QPB** may be equal to $\triangle$**ABC**.

23. Construct a triangle equal in area to a given triangle **ABC**, and having the perpendicular distance of a vertex from the opposite side equal to a given length **PQ**.

24. Construct a triangle equal in area to a given triangle **ABC**, having one of its sides along a given line **PQ**, and the opposite vertex at a given point **O**.

25. **To bisect a quadrilateral by a straight line drawn through one of its vertices.**

To bisect the quad. **ABCD** by a st. line drawn through **A**. Join **AC**, **BD**; bisect **BD** at **O**. Join **AO**, **CO**. Then quad. **AOCD** = $\frac{1}{2}$ quad. **ABCD**. Through **O** draw **OP** || to **AC** cutting **BC** in **P**. Join **AP**. Then **AP** bisects the quad. **ABCD**. The proof is left to the student. (Show that fig. **APCD** = fig. **AOCD**).

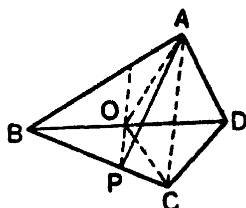


Fig. 289

Another method : Construct a △ equal in area to **ABCD**, with **A** for a vertex, **AB** for a side, and the side opposite to **A** lying along **BC** (see Prob 8, § 120). If **P** be the mid-point of that side of the △, which lies along **BC**, **AP** will bisect **ABCD**.

26. Bisect a trapezium by a straight line drawn through one of its vertices.

27. **PQRSTU** is the plan of a field; show how to replace the fence **PQRS** by a straight fence **PX**, **X** being some point in **TS** or **TS** produced, *without altering the area of the field.*

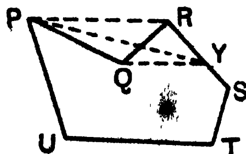


Fig. 290

[Join **PR**, draw **QY** || to **PR**, cutting **RS** in **Y**, then fig. **PYSTU** = fig. **PQRSTU**. You may now draw through **P** a line **PX** cutting **TS** (produced if necessary) at a point **X**, so that the fig. **PXTU** = fig. **PYSTU** = fig. **PQRSTU** [see note, § 120].

126. Theorem of Pythagoras. The relation between the squares on the sides of a right-angled triangle.

Fig. 291 on the margin, shows how the square on the hypotenuse **AB**, of an isosceles right-angled △ **ABC**, is divided into four right-angled triangles each equal to **ABC**. The squares on the equal sides **AC**, **BC**, contain two such triangles each. Thus the square on **AB** = the sum of the squares on **AC**, **BC**.

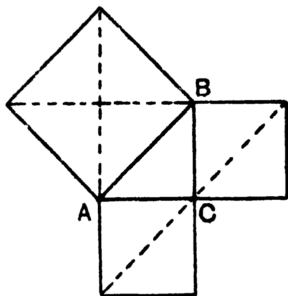
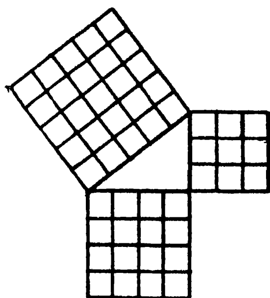


Fig. 291

Draw a right-angled triangle, the sides containing the right-angle being 3 and 4 units of length. Measure the hypotenuse, and see if the square on the hypotenuse = the sum of the squares on the sides containing the right-angle.


Fig. 292

Draw a number of right-angled triangles and in each case, measure the three sides, and calculate the areas of the squares upon them. Add together the areas of the two smaller squares and compare the sum with the area of the square on the hypotenuse.

Do these experiments lead you to suspect that *in a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the sides containing the right-angle ?*

THEOREM 25. [Euclid 1, 47.]

127. **In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.**

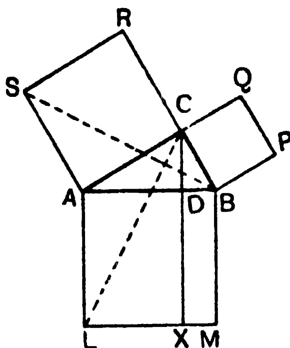


Fig. 293

△ **ABC** is right-angled at **C**.

The figures **AM**, **BQ**, and **CS** are squares described upon **AB**, **BC**, **CA** respectively,

To prove that sq. **AM** = *sq.* **CS** + *sq.* **BQ**.

From **C** draw **CX** ∥ to **AL** or **BM**. Join **BS**, **CL**.

Proof. Since ∠s **ACB**, **BCQ**, **ACR** are right-angles,

∴ **ACQ**, **BCR** are straight lines.

To each of the right-angles **LAB** and **SAC**, add ∠**BAC**,

∴ ∠**CAL** = ∠**SAB**.

Now in the $\triangle^s$ **CAL, BAS,**

AC = **AS,** (*sides of the same square*)

AL = **AB,** („ „ „)

$\angle$**CAL** = $\angle$**SAB.** *proved*

$\therefore$ the $\triangle^s$ **CAL, BAS** are congruent. (*Theor.* 4)

But rect. **AX** = 2 $\triangle$ **CAL.** ($\because$ they are on the same base **AL** and between the same $\|^s$ **AL, CX**)

and sq. **CS** = 2 $\triangle$ **BAS.** ($\because$ they are on the same base **SA** and between the same $\|^s$ **SA, RB**).

$\therefore$ rect. **AX** = sq. **CS.**

In a similar way, by joining **AP** and **CM**, it may be shown that rect. **BX** = sq. **BQ.**

Hence sq. **AM** = rect. **AX** + rect. **BX.**

= sq. **CS** + sq. **BQ.**

i.e. the square on **AB** = sum of the squares on **AC, BC.**

Note 1. This theorem was discovered by Pythagoras (B.C. 570—500) and is known as the **Theorem of Pythagoras.**

An Alternative Proof.—Below is given a proof which will show how the two smaller squares might be cut up in such a way, that these parts being pieced together in a certain manner would produce the square on the hypotenuse.

ABC is a $\triangle$ right-angled at **C.**

ALMC is the square on **CA.**

MQ is taken = **BC.**

The fig. **MP** is the square on **MQ**, it is equal to the square on **BC**, for **MQ** = **BC.**

R is a point in **LM**, so that **RL** = **MQ** = **BC.**

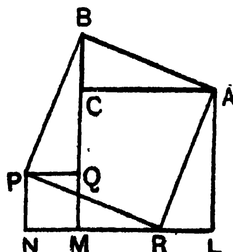


Fig. 294

Join **AR, RP, PB.** It may easily be shown that $\angle^s$ **ALR, PNR, BPQ, ABC** are all congruent,

and the figure **ABPR** is a square.

Now sq. on **BC**+sq. on **AC**=fig. **ACQPNLA**.

If from this figure we cut away the triangles **ARL** and **PNR** and place them in the positions **ABC** and **BPQ** we get the figure **ABPR** which is the square on **AC**, and it is equal in area to the figure **ACQPNLA**. Thus it is proved that

sq. on **AB**=*sq.* on **BC**+*sq.* on **CA**.

Note 2. The result established in this Theorem may be stated as follows :—

$$AB^2 = AC^2 + BC^2$$

If **AB**=p units of length, **BC**=q units and **CA**=r units, then

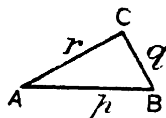


Fig. 295

sq. on **AB**=p^2 square units, *sq.* on **BC**=q^2 square units, and *sq.* on **AC**=r^2 square units ;

and we have

$$p^2 = q^2 + r^2. \qquad (1)$$

$$\text{or} \quad p = \sqrt{(q^2 + r^2)}. \qquad (2)$$

From (1) we get,

$$q^2 = p^2 - r^2. \qquad (3)$$

$$\text{and } r^2 = p^2 - q^2, \qquad (4)$$

So that $$q = \sqrt{(p^2 - r^2)}, \qquad (5)$$

and $$r = \sqrt{(p^2 - q^2)}. \qquad (6)$$

If the lengths of two sides of a rt $\angle$d triangle are known that of the third side may be calculated by one of the formulæ (2), (5), (6).

Note 3. If **CX** cuts **AB** in **D** (Fig. 293) we see that **CD** is $\perp$ to **AB**. It has been shown in course of the proof that

rect. **AX**=*sq.* on **AC**

and rect. **BX**=*sq.* on **BC**

i.e., $$AB.AD = AC^2$$

$$AB.BD = BC^2$$

EXERCISE LII.

1. Draw a right-angled triangle of sides 7 cm., 8 cm., calculate the length of the hypotenuse correct to a millimetre, and verify by measurement.

2. Draw a right-angled triangle the hypotenuse being 7 cm. and one side = 5 cm.; calculate the length of the other side and verify the result by measurement.

3. Calculate the hypotenuse of a right-angled triangle when the sides containing the right-angle are

(i) 5 in., 12 in. (ii) 6 cm., 8 cm. (iii) 30 ft., 16 ft. (iv) 100 yd., 150 yd.

4. Find the remaining side, given the hypotenuse and one side

(i) 10 cm., 8 cm. (ii) 13 in., 12 in.
(iii) 120 mm., 75 mm (iv) 25 ft., 15 ft.
(v) 3·5 in., 2·4 in.

5. Find the length of the diagonal of a rectangle whose sides are. (i) 4 in., 5 in (ii) 20 ft., 30 ft (iii) 150 yds., 250 yds (iv) 75 cm., 75 cm

6. A ladder 50 ft. long is placed against a vertical wall, with its foot on the level ground distant 15 ft. from the wall. To what height on the wall does the ladder reach?

7. A ladder just reaches to the top of a wall 30 ft. high, with its foot on level ground distant 8 ft from the wall. Find the length of the ladder.

8. **ABC** is any triangle and **AD** is drawn perpendicular from **A** to **BC**. Show that the difference of the squares on **BD**, **CD** is equal to the difference of the squares on **AB**, **AC**.

9. Show that the perpendicular drawn from the centre of a circle to any chord, bisects that chord

10. *If two chords of a circle are equidistant from the centre, they are equal.

11. Find the length of a chord of a circle of radius 10 ft. if the distance of the chord from the centre is 4 5 feet.

12. A gun whose range is 2 miles is situated at a distance

of 1·2 miles from a straight road. What length of the road is commanded by the gun?

13. **ABC** is an equilateral triangle of side 5 inches, find the distance of any vertex from the opposite side.

14. An isosceles triangle has its base = 10 cm., and the equal sides = 7 cm. each. Find the altitude and the area.

15. A vertical post 30 ft. high is held up by ropes each 50 ft. long; each rope is tied at one end to the top of the post, and at the other end to a peg in the ground. Do the pegs all lie in a circle? If so, what is the radius of the circle?

16. A rectangular field is 650 yards long and 400 yards broad. How many minutes will a person take to cross it diagonally, walking at the rate of 3 miles per hour?

17. In Ex. 16, what time is saved by cutting diagonally across the field instead of walking along the two edges?

18. **P, Q, R** are three towns. **P** is 10 miles north of **Q**, and $7\frac{1}{2}$ miles east of **R**. How far is **Q** from **R**?

19. Two persons start simultaneously from the same place. **A** walks due west at the rate of 4 miles per hour, and **B** due south at the rate of $3\frac{1}{2}$ miles per hour. How far apart will they be after 3 hours?

20. Find the length of the member **AB** of the girder shown in Fig. 296.

21. A room is 20 ft. long, 15 ft. wide, and 12 ft. high find the distance from a corner of the floor to the opposite corner of the ceiling.

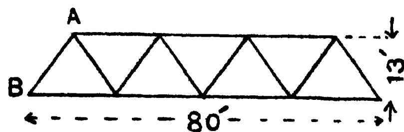


Fig. 296

22. **ABCD** is a rectangle and **O** is any point within it. Show that $OA^2 + OC^2 = OB^2 + OD^2$.

23. **ABC** is a triangle, and **O** is any point within it. Perpendiculars **OX, OY, OZ** are drawn to **BC, CA, AB**; show that $AZ^2 + BX^2 + CY^2 = AY^2 + CX^2 + BZ^2$.

24. If a quadrilateral **ABCD** has its diagonals at right angles to each other, show that $AB^2+CD^2=BC^2+AD^2$.

25. **ABC** is an equilateral triangle and **D** is the mid-point of **BC**, show that $AD^2=3\ BD^2=\frac{3}{4}\ AB^2$.

26. **ABC is a triangle with the ∠C = a rt. ∠, and ∠B = 60°. Prove that $AC^2=3BC^2$ and AB = 2BC.**

27. **ABC** is a triangle right-angled at **C**. Two points **P** and **Q** are taken on **CA**, **CB** respectively. Show that $AQ^2+BP^2=AB^2+PQ^2$.

28. Prove that a square is equal to half the square on a diagonal.

29. Prove that the sum of the squares on the sides of a rhombus is equal to the sum of the squares on its diagonals.

30. If an equilateral triangle and a square stand on the same base, the area of the triangle is $\frac{\sqrt{3}}{4}$ times the area of the square (see Ex. 25).

31. Show that the equilateral triangle described on the hypotenuse of a right-angled triangle is equal to the sum of the equilateral triangles described on the sides containing the right-angle. (Use Theor. 25 and Ex. 30).

32. Draw a square which shall be (i) twice as large as a given square, (ii) half as large as a given square .

33. Draw a square which shall be (i) three times as large as a given square, (ii) 4 times as large, (iii) 5 times as large, and so on.

34. Divide a straight line into two parts such that the square on one part may be double the square on the other part

35. Divide a straight line into two parts such that the square on one part may be three times as large as the square on the other part. (see Ex. 26)

36 Divide a straight line into two parts such that the sum of the squares on the two parts may be equal to a given square. When does the problem fail ?

37. Divide a straight line into two parts such that the difference of the squares on the two parts may be equal to a given square. When does the problem fail ?

THEOREM 26. [Euclid 1, 48.]

128. **If a triangle is such that the square on one side is equal to the sum of the squares on the other two sides, then the angle contained by these two sides is a right-angle.**

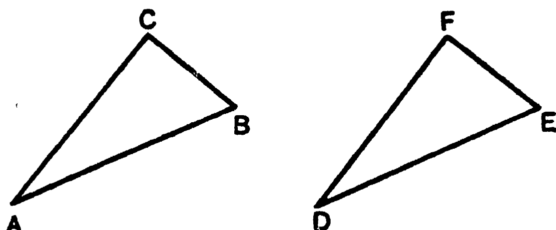


Fig. 297

ABC is a triangle in which $AB^2 = AC^2 + BC^2$.

To prove that $\angle ACB$ is a right angle.

Construct a $\triangle$ DEF in which $\angle F$ is a right angle and FD = CA and FE = CB.

Proof. Since $\angle DFE$ is a right angle, (*Constr.*)

$\therefore$ $DE^2 = DF^2 + EF^2$;

$= AC^2 + BC^2$, (*Constr.*)

$= AB^2$; (*Given*)

$\therefore$ DE = AB.

Now in the $\triangle^s$ ABC, DEF,

$\begin{cases} AC = DF, & (Constr.) \\ BC = EF, & (Constr.) \\ AB = DE, & (Proved) \end{cases}$

$\therefore$ the $\triangle^s$ are congruent;

$\therefore$ $\angle ACB = \angle DFE$.

But $\angle DFE$ is a right angle. (*Constr.*)

$\therefore$ $\angle ACB$ is a right angle.

Note. This theorem suggests a method for drawing the perpendicular to a given line from a given point **A** in it. Since $5^2 = 3^2 + 4^2$, if you construct a triangle whose sides are 3, 4, 5 units of length respectively, the angle opposite to the longest side (5) will be a right angle. On the given line mark off a length **AC**=3 cm. With centre **A** and radius 4 cm. describe a circle, and with centre **C** and radius 5 cm. describe another circle. Let the two circles cut at a point **B**. Then **AB** is perpendicular to the given line. This method for the construction of a right angle was used by the Egyptians even two thousand years before Christ was born.

Ex. 1. Are the triangles whose sides are given below, right-angled ?

(i) 3, 5, 7 (units of length) (ii) 5. 12, 13 (iii) 8, 17, 15 (iv) 5. 8, 9.

Ex. 2. Show that if the sides of a triangle are x^2+1, x^2-1, $2x$, the triangle is right-angled.

Giving to x values 2, 4, 6, obtain three sets of lengths which will determine the sides of certain right-angled triangles

Ex. 3. If the triangle whose sides are of lengths a, b, c, is right-angled, show that the triangle whose sides are of lengths ma, mb, mc (where m is any positive number) is also right-angled.

129. Two applications of the Theorem of Pythagoras.

1. Graphical method for finding approximate values of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, etc.

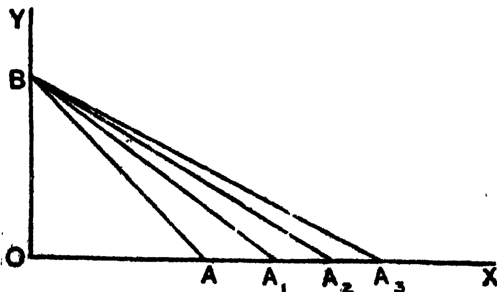


Fig. 298

Take two lines **OX**, **OY** at right angles, and cut off **OA**, **OB** each = 1 inch. Join **AB**, then

$$AB^2 = OA^2 + OB^2 = 1 \text{ sq. in.} + 1 \text{ sq. in.}$$
$$= 2 \text{ sq. in.}$$
$$\therefore AB = \sqrt{2} \text{ inch.}$$

If you measure **AB** in inches and decimals of an inch, the measure will give an approximate value of $\sqrt{2}$. Now mark off $OA_1 = AB$.

Then $A_1B^2 = OB^2 + OA_1^2 = 1$ sq. in + 2 sq in

$$= 3 \text{ sq. in}$$
$$\therefore BA_1 = \sqrt{3} \text{ in;}$$

thus the measure of BA_1 in inches and decimals of an inch will give an approximate value of $\sqrt{3}$.

To get $\sqrt{4}$, you mark off $OA_2 = BA_1$, the measure of BA_2 will give $\sqrt{4}$.

Proceeding in this manner you can obtain in succession $\sqrt{5}$, $\sqrt{6}$, ..., indeed square roots of all integers.

II. Calculation of the area of a triangle, when the lengths of the three sides are given.

Let **ABC** be a triangle, in which **BC**, **CA**, **AB** are a, b, c units of length respectively.

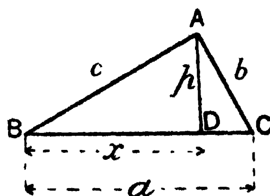


Fig. 299

Let **AD** be perpendicular to **BC**.

If **AD** = p units of length, the area of the $\triangle$ **ABC** = $\frac{1}{2}$ pa square units.

Now a, b, c are known, because the lengths of the sides are, by hypothesis, given The length of **AD** (*i.e.*, p) is not given, how to find p by *calculation*?

Let **BD** = x units of length, Then **CD** = $(a-x)$ units of length.

Since $\angle$**BDA** is a right angle, by the Theorem of Pythagoras we have.

$$\mathbf{AD^2 = AB^2 - BD^2},$$

i.e., $p^2 = c^2 - x^2$; (1)

from the right-angled ACD we have

$$\mathbf{AD^2 = AC^2 - CD^2},$$

i.e., $p^2 = b^2 - (a-x)^2$ (2)

From (1) and (2) we have

$$c^2 - x^2 = b^2 - (a-x)^2$$

or $c^2 - x^2 = b^2 - a^2 + 2ax - x^2$

$\therefore \quad 2ax = a^2 - b^2 + c^2$

$$\therefore \quad x = \frac{a^2 - b^2 + c^2}{2a} \qquad (3)$$

Now by (1), $p^2 = c^2 - x^2$

$$= c^2 - \left(\frac{a^2 - b^2 + c^2}{2a}\right)^2, \text{ [substituting for } x \text{ from (3)]}.$$

$$= \frac{4a^2c^2 - (a^2 - b^2 + c^2)^2}{4a^2}$$

$$= \frac{(2ac)^2 - (a^2 - b^2 + c^2)^2}{4a^2}$$

$$= \frac{(2ac + a^2 - b^2 + c^2)(2ac - a^2 + b^2 - c^2)}{4a^2}$$

$$= \frac{[(a+c)^2 - b^2][b^2 - (a-c)^2]}{4a^2}$$

$$= \frac{(a+c+b)(a+c-b)(b+a-c)(b-a+c)}{4a^2}$$

$$= \frac{4s(s-b)(s-c)(s-a)}{a^2}$$

where $2s = a + b + c$ (i.e. the perimeter of the $\triangle$)

$$p = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$$

Now the area of the $\triangle$ABC $= \frac{1}{2}ap$ sq. units.

$$= \sqrt{s(s-a)(s-b)(s-c)} \text{ sq. units,}$$

where s is the semi-perimeter of the triangle.

Ex. Calculate the areas of the triangles whose sides are :

(i) 20 in., 30 in., 40 in.

(ii) 55 ft., 75 ft., 100 ft.

(iii) 60 yds. 85 yds., 49 yds.

CHAPTER X.

LOCI.

130. The word '*locus*' is the Latin name for 'place'. We shall now see how this term is used in Geometry.

An illustration.

Through a given point **O**, draw a number of straight lines, and cut off lengths OP_1, OP_2, OP_3,..., each equal to $\frac{1}{2}$ inch.

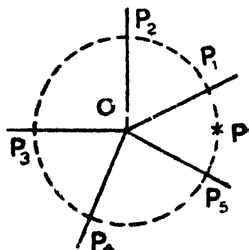


Fig. 300

The points P_1, P_2, P_3, etc. all lie on a circle whose centre is **O** and radius $= \frac{1}{2}$ inch. Indeed *every point whose distance from* **O** *is* $\frac{1}{2}$ *inch, lies on this circle, and every point on this circle is at a distance of* $\frac{1}{2}$ *inch from* **O**.

We express this result by saying that the locus of all points which are at a distance of $\frac{1}{2}$ inch from a specified point **O**, is a circle with centre at **O**, and radius $\frac{1}{2}$ inch.

What is the locus of points which are at a distance of (i) 2 inches from **O**, (ii) 8 cm. from **O** ?

Another illustration. Take a line **AB** and suppose it to be unlimited in length. Mark any points N_1, N_2,...

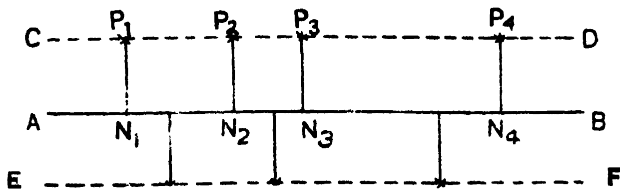


Fig. 301

in it. From these points draw ⊥s to **AB**, and take lengths N_1P_1, N_2P_2, N_3P_3...each equal to 1 cm.

The points P_1, P_2,... all lie on a straight line **CD** parallel to **AB**,* the perpendicular distance between the two lines being 1 cm. And *all points which are above the line* **AB**, *and at a distance of 1 cm. from it lie on* **CD**, *and every point on* **CD** *is at a distance of 1 cm. from* **AB**.

In the same way, all points which are *below* the line, **AB**, and at a distance of 1 cm. from it, lie on the line **EF**, also parallel to **AB** and 1 cm. from it., but lying below **AB**.

Thus (i) the **locus** of all points lying above the line **AB**, and 1 cm. from it, is the straight line **CD** (ii) the locus of all points lying below **AB** and 1 cm. from it, is the straight line **EF**, and (iii) the locus of all points above or below **AB**, 1 cm. from **AB**, is **the pair of lines CD, EF**.

In the first of the illustrations considered above, if we take a point **P** at a distance of ½ inch from **O** (Fig. 300), and suppose this point to move so as to remain always at this distance (½ inch) from **O**, it will trace out the circle with centre **O**, and radius ½ inch, and in

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* N_1P_1, N_2P_2, N_3P_3 being all ⊥...to **AB** are parallel.

Join P_1P_2; Since N_1P_1, N_2P_2 are equal and parallel, ∴ P_1P_2 is ∥ to **AB** (Theor. 19). Similarly by joining P_1P_3 we can show that P_1P_3 is ∥ to **AB**; in the same way P_1P_4 etc. are all ∥ to **AB**.

course of its motion will pass through the positions $P_1, P_2, P_3, \dots$

We express this result also by saying that the locus of a point **P** which *moves* so as to be always at a distance of half an inch from **O**, is the circle with centre **O** and radius ½ inch.

Ex. 1. What is the locus of points on the ground at a distance of 50 feet from the foot of a certain tree.

Ex. 2. What is the locus of all points on the floor, 2 feet from a certain wall of the room.

Ex. 3. How many points can you mark on the paper which are at a distance of 1 inch from a given line **XY**. Do all these points lie on a line or a pair of lines? In case the points lie on a line or a pair of lines, are all the points of this line or pair of lines at a distance of 1 inch from **XY**?

Ex. 4. Draw a straight line on your paper and construct the locus of points lying on one side of the line and distant 0·7 in., from it

Ex. 5. **AB** is a given straight line and **A** is a given point in it. Find the locus of the point **P**, such that ∠**PAB** is always 50°.

Ex. 6 What is the locus of the moving end of a hand of a clock?

Ex. 7. A point is marked on a blackboard, what is the locus of this point if (i) the board slides along two vertical grooves, (ii) if it turns round a horizontal axis?

Hence the lines P_1P_2, P_1P_3, P_1P_4, .. must be the same line, for through the point P_1 there passes only one straight line which is parallel to **AB**. (Playfair's Axiom, see foot-note p. 192).

Ex. 8. If a wheel rolls on the ground along a straight line, what is the locus of its centre ?

Ex. 9. If a straight rod makes one complete turn round one of its ends, what is the locus of its mid-point ?

Ex. 10. **O** is a fixed point, and **P** is any point such that the distance **OP** $>$ 1 inch. How many points (**P**) are there whose distance from **O** is greater than 1 inch ? Do they all lie on a certain line ? Have the points **P** any locus ?

From the above examples you will come to realise that **the locus of all points satisfying a certain specified condition** (e. g. being at a given distance from a fixed point, or being equidistant from two fixed points) **is the line on which all such points lie and which contains no point that does not satisfy the specified condition.** That is to say, the locus is the line consisting of all the points satisfying the specified condition, and *no other points.*

If the specified condition be such that the points satisfying it are not all confined to a definite line (or a definite number of lines), the points do not give us a *locus* in the sense in which we understand this term. Ex. 10 is a case where no definite locus is obtained ; indeed all points lying outside the circle of centre **O** and radius 1 inch, satisfy the specified condition, and obviously these points do not determine a definite line (or a definite number of distinct lines).

In fact any point **P** which is outside the circle whose centre is **O** and radius 1 inch satisfies the condition, **OP**$>$1 in., just as any point **P** inside the circle satisfies the condition **OP** $<$ 1 in. It is thus seen that the statement **OP**$>$ 1 in defines the surface outside that circle, the statement **OP**$=$1 in. defines the circumference of that circle, and the statement **OP** $<$1 in. defines the surface within the circle.

131. How to trace a locus.

If we can devise a mechanical contrivance by means of which a point (e.g. a pencil-point) may be made to move in a continuous manner, so as to satisfy a specified condition in all its positions, the line traced out by the point will be the required locus. Such mechanical devices as ruler and compasses will enable you to draw such loci as are straight lines or circles. If the locus happens to be a curve other than a circle, and if you want to obtain it from the motion of a point, you will have to devise an instrument other than compasses, for this purpose. Mechanical devices are not possible in all cases.

The question now arises—How are we to trace a locus when we have no mechanical device for it.

Going back to the second illustration in § 128, we see that by taking a series of points N_1, N_2,...on **AB** and erecting perpendiculars to **AB** (Fig. 301) at these points and taking lengths N_1P_1, N_2P_2,... each = 1 cm., we may obtain a series of points P_1, P_2,...all lying above the line **AB** and at a distance of 1 cm. from it. If we had taken care to mark the points N_1, N_2,...at close intervals the points P_1, P_2, P_3,... would also be at close intervals, and joining these points by a continuous line you would obtain a portion of the locus, approximately.

This method is known as '**plotting the locus**', and may be used in all cases (even when mechanical devices are available). By plotting the points at sufficiently close intervals, you can draw the locus as accurately as you desire.

Ex 1. **To plot the locus of a point P the sum of whose distances, from two fixed points A and B is constant.**

As a particular case let us take **AB** = 1 inch, and **P** to be such

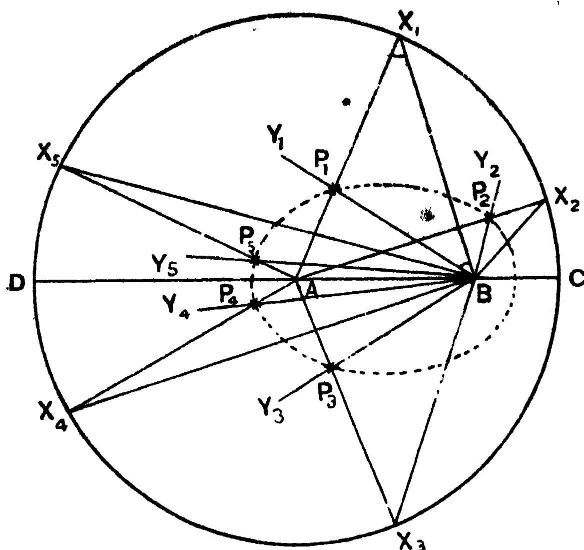


Fig. 302

that $\mathbf{AP+BP}=1\cdot5$ inches. With centre **A** and radius $=1\cdot5$ in., draw a circle. Mark any point $\mathbf{X}_1$ on the circle and join $\mathbf{AX}_1$, $\mathbf{BX}_1$. At **B** make $\angle\mathbf{X}_1\mathbf{BY}_1=\angle\mathbf{AX}_1\mathbf{B}$ and let $\mathbf{BY}_1$ cut $\mathbf{AX}_1$, in $\mathbf{P}_1$. Then $\mathbf{P}_1$ is a point of the locus.

For, $\mathbf{BP}_1=\mathbf{P}_1\mathbf{X}_1$ ($\because\ \angle\mathbf{P}_1\mathbf{X}_1\mathbf{B}=\angle\mathbf{P}_1\mathbf{BX}_1$ by construction),

$\therefore\ \mathbf{AP}_1+\mathbf{BP}_1=\mathbf{AP}_1+\mathbf{P}_1\mathbf{X}_1=\mathbf{AX}_1=1\cdot5$ inches.

Similarly by taking other points $\mathbf{X}_2$, $\mathbf{X}_3$,...on the circumference and proceeding as before you can find any number of points $\mathbf{P}_2$, $\mathbf{P}_3$,...of the locus. By joining these points by a continuous line you obtain the required locus as shown by the dotted line (Fig. 302). Such a locus (oval in shape) is called an **Ellipse**. Show that the locus (Fig. 302) passes through the mid-points of **BC**, **BD**.

We may have a mechanical device for the construction of this locus (the ellipse), as shown bellow.

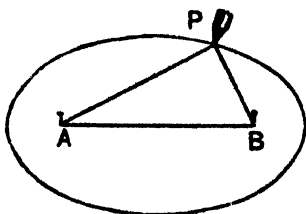


Fig. 303

Take a piece of thread 1·5 inches, long with its ends fastened to the fixed points **A**, **B**. Hold the point of a pencil, against the thread as shown in Fig. 303, and move the pencil, keeping the piece of thread always taut Since **AP+BP**= 1·5 inches in all positions of the pencil-point, the line traced out by the pencil-point is the required locus.

Ex. 2. **B** and **C** are two fixed points, plot the locus of a point **A** such that ∠**BAC** may be equal to a given angle. [See § 104, Fig. 239].

Ex. 3 **AB** is a fixed straight line, and **O** a fixed point outside it. **O** is joined to any point **Q** in **AB**, and **OQ** is produced to **P**, such that **QP**=$\frac{1}{8}$ **OP**. Plot the locus of **P**.

Ex. 4. **AB** is a fixed straight line, and **X** any point in it: Through **X**, **XP** is drawn such that **XP**=4 cm., and ∠**PXA**=50°. Plot the locus of **P**.

Ex. 5. A straight rod **PQ** of fixed length slides between two fixed rods **OA**, **OB** at right angles to each other. Find the locus of the mid-point of **PQ**. (Fig. 304).

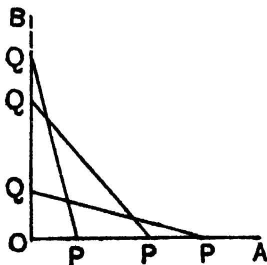


Fig. 304

Ex. 6. On a fixed line **BC** as base, triangles **BAC** are described such that in each case **AB**=2**AC**. Plot the locus of **A**.

Ex. 7. **A** and **B** are two fixed points 5 cm. apart. Plot the locus of a point **P** such that **AP+BP** =8 cm. [See Ex. 1]

Ex. 8. How would a gardener mark out an elliptic bed?

Ex. 9. Draw a circle of radius 1 inch, and take a fixed point **O**

0·6 in. from the centre. **P** is a movable point on the circumference; **OP** is joined and produced to **Q** such that **OQ=2OP**. Find a sufficient number of positions of **Q** and plot its locus.

Ex. 10. **AX** is a fixed rod, and **A** a fixed point in it. A movable rod **AB** is hinged at **A** to the fixed rod **AX**, and at **B** to another movable rod **BC**; the end **C** of the rod **BC** is fastened to a small ring which slides along the rod **AX** Plot the locus of the mid-point of **BC**.

Ex. 11. **A** and **B** are two fixed points. Plot the locus of points equidistant from **A** and **B**.

Ex. 12. **OX**, **OY** are two straight lines at right angles Plot the locus of points within the ∠**XOY**, which are equidistant from **OX** and **OY**.

Ex. 13. In Ex. 12, plot the locus of all points within the ∠**XOY** whose distance from **OX** is double the distance from **OY**.

Ex. 14. **Plot the locus of a point which is equidistant from a fixed straight line AB and a fixed point O outside AB.** [Take any point **X** in **AB**, and draw **XQ** ⊥ to **AB**, on the same side of **AB** as **O** Join **OX**, and at **O** make ∠**XOR**=∠**OXQ**. Let **XQ** and **OR** meet in the point **P**. Then **P** is a point on the locus By taking other points (**X**) on **AB**, and proceeding as before you may find as many points (**P**) of the locus as you like Joining these points by a continuous line you can obtain the locus. Such a locus is called a **Parabola.**

Ex. 15 **Find the locus of a point P such that AP−PB=a given length, A and B being two fixed points.** [With centre **A** and radius=given length, draw a circle. Take any point **X** on the circumference. Join **AX** and produce it to **Y**. Join **BX**, and at **B** make ∠**XBR**=∠**BXY**. Let **BR** cut **AY** in **P**. Then **P** is a point of the locus, for you can prove that **AP−BP**=given length. Such a locus is called a **Hyperbola.**]

THEOREM 27.

132. The locus of a point equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.

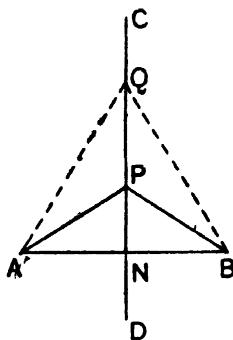


Fig. 305

A and B are two fixed points.

To prove that the locus of a point equidistant from A and B is the perpendicular bisector of AB ;

i. e. *to prove that* (i) any point equidistant from A and B, lies on the perpendicular bisector of AB, and (ii) any point on the perpendicular bisector of AB is equidistant from A and B.

(i) Let P be *any* point equidistant from A and B, so that AP = BP. Then P is on the perpendicular bisector of AB.

Proof. Join **AB**, and let **N** be the mid-point of **AB**.

Join **PN**.

In the $\triangle$s **ANP**, **BNP**

{ **AP** = **BP**, (*Given*)
{ **AN** = **NB**,
{ **PN** is common ;

$\therefore$ the $\triangle$s are congruent : (*Theor.* 7)

$\therefore$ $\angle$**ANP** = $\angle$**BNP**,

$\therefore$ **NP** is $\perp$ to **AB** ;

i. e. **P** lies on the perpendicular bisector of **AB**.

(ii) Let **Q** be any point on the perpendicular bisector of **AB**, i. e. the line **CND**. Then **Q** is equidistant from **A** and **B**.

Proof. In the $\triangle$s **ANQ** and **BNQ**,

{ **AN** = **NB**,
{ **NQ** is common,
{ $\angle$**ANQ** = $\angle$**BNQ** (being right $\angle$s),

$\therefore$ the $\triangle$s are congruent. (*Theor.* 4)

$\therefore$ **AQ** = **BQ**.

From (i) and (ii) it follows that the locus of a point equidistant from **A** and **B** is the perpendicular bisector of **AB**.

Note. In proving that a certain line (straight or curved) is the locus of a point satisfying a certain specified condition, two things must be shown :

(i) that all points satisfying the specified condition lie on this line ;

(ii) that every point on the line satisfies the specified condition.

THEOREM 28.

133. The locus of a point which is equidistant from two given intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.

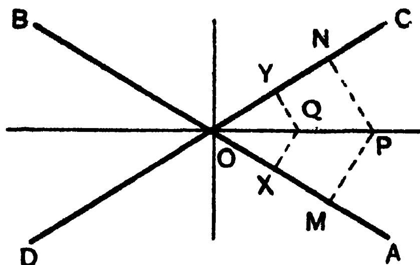


Fig. 306

AOB and COD are two intersecting straight lines.

To prove that the locus of a point equidistant from AOB and COD is the pair of straight lines which bisect the angles between the lines AOB, COD ;

i. e. *to prove that* (i) any point equidistant from AOB, COD lies on one or other of the bisectors of the angles between AOB, COD, and (ii) any point on any of the bisectors of the angles between AOB, COD is equidistant from AOB, COD.

Proof. (i) Let P be any point within ∠AOC (or ∠BOD), such that its perpendicular distances PM, PN from AOB, COD respectively, are equal;

then P is on the bisector of ∠AOC (or ∠BOD).*

* Note that the same straight line bisects a pair of vertically opposite angles.

In the △s **OPM, OPN,**

$\begin{cases} MP = NP, & (Given) \\ OP \text{ is common}, & \\ \angle s\ OMP, ONP \text{ are right } \angle s; & \end{cases}$

∴ the △s are congruent, (*Theor.* 18)

∴ ∠**MOP** = ∠**NOP**,

That is, **P** lies on the bisector of ∠**AOC** (or ∠**BOD**). Similarly it might be shown that if **P** were within ∠**AOD** (or ∠**BOC**), and equidistant from **AOB** and **COD**, it would lie on the bisector of ∠**AOD** (or ∠**BOC**).

(ii) Let **Q** be a point on the bisector of ∠**AOC** (or ∠**BOD**), then **Q** is equidistant from **AOB** and **COD**. Draw **QX, QY** perpendicular to **AOB, COD** respectively.

In the △s **OXQ, OYQ,**

$\begin{cases} \angle XOQ = \angle YOQ, & (Given) \\ \angle OXQ = \angle OYQ \text{ (being rt. } \angle s), & \\ OQ \text{ is common}; & \end{cases}$

∴ the △s are congruent. (*Theor.* 17)

∴ **QX** = **QY**,

That is, **Q** is equidistant from **AOB, COD.**

Similarly it may be shown that any point on the bisector of ∠**AOD** (or ∠**BOC**) is equidistant from **AOB, COD.**

From (i) and (ii) it follows that the locus of a point equidistant from **AOB, COD,** is the pair of straight lines which bisect the angles between **AOB, COD.**

134. Intersection of loci. The method of Loci.

In § 39, in order to find a point distant 1·5 in. from **A** and 2 in. from **B**, you drew two circles (Fig. 106) one

with centre **A** and radius 1·5 in., another with centre **B** and radius 2 in. The intersections of these two circles gave you the required points. You have employed what may be called the **method of intersection of loci.** The first circle is the locus of all points distant 1·5 in. from **A**, the second circle is the locus of all points distant 2 in. from **B**. The points common to these two loci must each be 1·5 in. from **A**, and 2 in. from **B**.

Draw two straight lines **AOB**, **COD** and suppose them to be unlimited in length. *How are you to find a point which is 1 in. from* **AOB** *and 1·5 in. from* **COD** ?

The locus of all points 1 inch from **AOB** is the pair of lines ∥ to **AOB**, on opposite sides of, and 1 inch from it (see § 130, second illustration). In the same way, the locus of all points 1·5 inch from **COD** is the pair of lines ∥ to **COD** on opposite sides of, and 1·5 in. from it. These two loci intersect in four points, each of which is 1 in. from **AOB**, and 1·5 in. from **COD**.

EXERCISE LIII.

1. Draw two straight lines **XOX′** and **YOY′** at right angles, and consider them to be unlimited in length. Find a point (or points) distant 3 cm. from **XOX′**, and 4 cm. from **YOY′**.

2. Draw two straight lines **XOX′**, and **YOY′** such that ∠**XOY** = 60°. Find a point (within ∠**XOY**) which is 0·7 in. from **XOX′** and 0·5 in. from **YOY′**.

3. Draw a straight line **AB** and mark a point **O**, 2 cm. from **AB**. Find a point (or points) in **AB** distant 3·5 cm. from **O**.

4. In Ex. 3, find a point (or points) distant 1 cm. from **AB** and 3·5 cm. from **O**. How many such points do you get ?

5. In Ex. 3 find a point (or points) distant 3 cm. from **AB** and 3·5 cm. from **O**. How many such points do you get ?

6. Find a point (or points) equidistant from two given points, **A** and **B**, and at a specified distance from a third given point **C**. How many such points do you get? Discuss all possible cases that may arise.

7 **ABC** is a triangle; find a point in **BC** (or **BC** produced either way) equidistant from **A** and **B**.

8 Find a point (or points) equidistant from two given intersecting lines, and at a specified distance from a given point. How many such points do you get? Discuss all possible cases that may arise.

9. Find a point (or points) equidistant from two given points, and at a specified distance from a given straight line.

10. **A** and **B** are two points on the circumference of a circle. Find a point (or points) on the circumference equidistant from **A** and **B**.

11. **AB** and **AC** are two chords of a circle. Find a point (or points) on the circumference equidistant from **AB**, **AC**.

12. **ABC** is a triangle. Find a point in the side **BC**, equidistant from **AB**, **AC**.

13 **ABC** is a triangle, find a point in **BC** (or **CB**) produced equidistant from **AB**, **AC**. If **AB** = **AC** do you get any such point?

14. Find a point (or points) equidistant from two given lines and at a specified distance from a third given line, no two of the given lines being parallel

How many such points do you get? Discuss all possible cases that may arise.

15. **A, B, C are three points not in one straight line. Find a point equidistant from A, B, C. (How many such points do you get?) Hence show that the perpendicular bisectors of the sides of a triangle are concurrent.**

16. **ABC is any triangle. Find a point within the triangle equidistant from the three sides. Hence show that the internal bisectors of the angles of a triangle are concurrent.**

17. **ABC** is any triangle. Find all the points equidistant from the lines **BC, CA, AB.** (How many points do you get ?). Hence prove the concurrence of the following sets of three lines:

(i) the internal bisector of ∠**A**, and the external bisectors of ∠s **B, C.**

(ii) the internal bisector of ∠**B**, and the external bisectors of ∠s **C, A.**

(iii) the internal bisector of ∠**C**, and the external bisectors of ∠s **A, B.**

Note. The case of the internal bisectors of the ∠s **A, B, C** has been noticed in Ex. 16.

18. To construct a triangle **ABC**, given the base **BC**, ∠**B**, and the altitude **AD.**

19. To construct a triangle **ABC**, given **AB, AC** and the altitude **AD**.

20. Given two pairs of intersecting straight lines, find a point which is equidistant from the first pair, and also equidistant from the second pair. How many such points are there?

21. On a given base construct a triangle of given altitude, and having the vertex on a given straight line.

EXERCISE LIV.

(on Loci.)

1. Find the locus of the centres of circles which pass through two given points.

2. **AB** is a fixed straight line, and **O** is a fixed point outside it. **Q** is a movable point on **AB**. Find the locus of the mid-point of **OQ.**

3. **A** and **B** are two fixed points; find the locus of a point **P** such that the area of the triangle **APB** may have a given value.

4. Two fixed lines **AB**. **AC** intersect in **A**; **B** and **C** are fixed points. Show that the locus of a point **P** such that the area of the quadrilateral **ABPC** may have a given value, is a straight line.

5. **AB** is a fixed straight line, and **O** a point out-side it. **Q**, a movable point on **AB**, is joined to **O** and **QO** is produced to **P**

such that **OP**=**QO**, show that the locus of **P** is a straight line parallel to **AB**.

6. **AB** and **CD** are two fixed parallel lines. **P** and **Q** are movable points on **AB**, **CD** respectively; show that the locus of the mid-point of **PQ** is a straight line parallel to **AB** (or **CD**).

7. **ABCP** is a convex quadrilateral. **A**, **B**, **C** are fixed points, and **P** a movable point such that **BP** is bisected by **AC**. Find the locus of **P**.

8. On a given line **AB** as base parallelograms of a given area are constructed; find the locus of the intersections of the diagonals.

9. **OX** and **OY** are two fixed straight lines at right angles. Find the locus of a point **P** within angle **XOY**, the sum of whose distances from **OX**, **OY** is equal to a given length.

10. **OX** and **OY** are two fixed straight lines at right angles. Find the locus of a point **P** within angle **XOY**, whose distance from **OX** exceeds the distance from **OY** by a given length

11 If a rod slides between two rods held at right angles to each other, show that the locus of its mid-point is a certain quadrant of a circle. [See Ex. 5, (XXXIX)].

12. Right-angled triangles are constructed on a given line **AB** as hypotenuse, and on the same side of it. Show that the locus of the vertices (opposite to the hypotenuse) is a certain semi-circle. [See note, § 105.]

13. **A** is a fixed point, and **Q** a movable point, both lying on the circumference of a fixed circle of centre **O**; show that the locus of the mid-point of **AQ** is the circle with **OA** for a diameter.

14. **B** is a fixed point outside a given circle of centre **O**, and **Q** is a movable point on the circumference. Show that the locus of the mid-point of **BQ** is a circle whose centre is the mid-point of **OB**.

15. **O** is a given point, and **AB** a given straight line. **Q** is a movable point in **AB**; **OQ** is joined, and a square **OQRP** is described on **OQ** in a specified sense. Show that the locus of **P** is a straight line perpendicular to **AB**.

[Draw **ON** ⊥ to **AB**, on **ON** describe a square **ONML** in the same sense as the square **OQRP** stands on **OQ**. Join **LP**.

Prove that the ∆s **OPL** and **ONQ** are congruent, so that ∠**OLP** is a right angle. Hence the locus of **P** is the line through **L** at right angles to **OL** and hence perpendicular to **AB**.]

16. **OX** and **OY** are two fixed straight lines at right angles. **P** and **Q** are two movable points on **OX** and **OY** respectively, such that **OP+OQ** is constant. Find the locus of the mid-point of **PQ**.

[The sum of the distances of the mid-point of **PQ** from **OX** and **OY** is constant; use Ex. 9.]

17. **OX** and **OY** are two fixed straight lines, at right angles. **P** and **Q** are two movable points on **OX**, **OY** respectively, such that **OP**−**OQ**=a given length. Find the locus of the mid-point of **PQ**.

18. **AB** is a fixed straight line, and **O** a fixed point outside it. **Q** is a movable point on **AB**; **OQ** is joined, and a line **OX** is drawn through **O**, such that ∠**QOX** is equal to a given angle in magnitude, and is in a given sense[1]. From **OX**, **OP** is cut off equal to **OQ**. Find the locus of **P**.

[Drop **ON** ⊥ to **AB**, and draw **OM**=**ON**, making ∠**NOM** equal to the given angle and in the specified sense. Thus ∠s**NOM**, **QOP** are equal. It may easily be proved that ∆s **OMP**, **ONQ** are congruent, and consequently ∠**OMP**=∠**ONQ**=a right angle. Thus the locus of **P** is the line through the point **M**, perpendicular to **OM**.][2]

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1. That is to say, if the line **OQ** were turned round the point **O** in the specified sense (clockwise, or anti-clockwise, as the case may be) through an angle equal to the given angle, it would come into coincidence with the line **OX** in direction.

2. The result may be arrived at in a more interesting way. The points **P** are the new positions of the points **Q**, obtained by rotating the line **AB** about an axis through **O**, (perpendicular to the plane of the paper on which the figure is supposed to be drawn) through an angle equal to the given angle, in the specified sense. Note that Ex. 15 is a particular case of Ex. 18.

CHAPTER XI.

MISCELLANEOUS THEOREMS AND PROBLEMS.

135. On the concurrence[1] of straight lines in a triangle.

I. The perpendicular bisectors of the sides of a triangle are concurrent.

L, M, N are the mid-points of the sides **BC, CA, AB** of the △**ABC**. At **M** and **N** perpendiculars are drawn to **AC, AB** respectively; let them meet in **O**. If we can show that **O** lies on the perp. bisector of **BC**, the theorem is established.

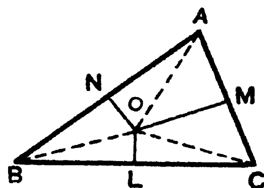


Fig. 307

Proof. Since **OM** is perp. bisector of **AC**.

OC = OA; [*Theor.* 27]

for a similar reason, **OB = OA**,

∴ **OB = OC**;

∴ **O** is on the perp. bisector of **BC**, [*Theor.* 27]

(that is, **LO** is ⊥ to **BC**).

Note. The theorem may be established directly without assuming Theor. 27; for, △s **AOM, COM** are congruent by Theor. 4, ∴ **OC = OA**. Similarly from the congruence of the △s **AON, BON, OB = OA**; ∴ **OB = OC**. Now by Theor. 7, the △s **OBL, OCL** are congruent, ∴ ∠**OLB** = ∠**OLC** = a right-angle.

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[1] See Note 1, page 9.

The point **O** where the perp. bisectors of **BC, CA, AB** meet, is equidistant from the three points **A, B, C**. Hence a circle drawn with centre **O** and radius **OA** will pass through **A, B, C**. This circle is said to be **circumscribed** about the triangle **ABC**, and is called the **circum-circle**. The radius of the circle is called the **circum-radius**, and the centre **(O)**, the **circum-centre** of the triangle.

II. **(a) The internal bisectors of the angles of a triangle are concurrent.**

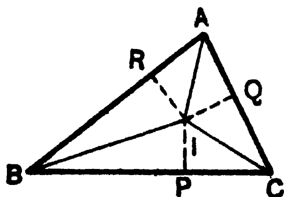


Fig. 308

Let the bisectors of $\angle$s **ABC, ACB**, meet in **I**.

If we can show that **I** lies on the bisector of $\angle$**BAC**, the theorem is established. From **I** draw **IP, IQ, IR** $\perp$ to **BC, CA, AB** respectively.

Proof. Since **BI** is the bisector of $\angle$**ABC**,

$\therefore$ **IR=IP** ; (*Theor.* 28)

for a similar reason, **IQ=IP**

$\therefore$ **IQ=IR**

$\therefore$ **I** is on the bisector of $\angle$**BAC** (*Theor.* 28)

(i.e., **AI** is the bisector of $\angle$**BAC**).

Note. As in I, the theorem may be established directly without assuming Theor. 28.

The point **I** where the internal bisectors of the angles of the triangle meet, is equidistant from the three sides. The circle drawn with centre **I** and radius **IP** will pass through **P, Q, R**.

II. (b) **The internal bisector of one angle of a triangle, and the external bisectors of the other two angles are concurrent.**

Let the *external* bisectors of ∠s **ABC**, **ACB** meet in I_1 and let I_1X, I_1Y, I_1Z be drawn ⊥ to **BC**, **CA**, **AB** respectively. As in II (a) we have $I_1Z = I_1X$

and $I_1Y = I_1X$,

∴ $I_1Y = I_1Z$.

∴ I_1 is on the internal bisector of ∠**BAC**. [*Theo.* 28]

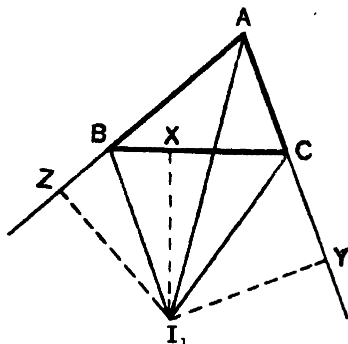


Fig. 309

Hence the internal bisector of ∠**A** and the external bisectors of ∠s **B**, **C** of the △**ABC** meet in the same point I_1.

Note. Since $I_1X = I_1Y = I_1Z$, the circle described with centre I_1 and radius I_1X will pass through **X**, **Y**, **Z**. It may be similarly shown that the internal bisector of ∠**B** and the external bisectors of ∠s **C**, **A**, are concurrent; also the internal bisector of ∠**C** and the external bisectors of ∠s **A**, **B** are concurrent.

III. **The three medians[1] of a triangle are concurrent.**

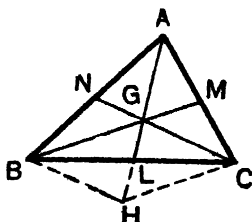


Fig. 310

Let the medians **BM**, **CN** of the △ **ABC** meet in **G**. Join **AG** and produce it to meet **BC** in **L**. If we can show that **L** is the mid-point of **BC**, the theorem is established.

Through **B** draw **BH** ∥ to **NC** to meet **AG** produced in **H**. Join **CH**.

1. See Note, Ex. 28, page 143.

Proof. In the △ **ABH**, **N** is the mid-point of **AB** and **NG** is ∥ to **BH**. ∴ **G** is the mid-point of **AH**. [see § 102 (a)]

Now in the △ **ACH**, **M** is the mid-point of **AC**, and **G** is the mid-point of **AH** (proved),

∴ **MG** is ∥ to **CH**. [see § 102, (b)]

Hence the figure **BGCH** is a parallelogram. [see § 97]

But the diagonals of a parallelogram bisect one another.

∴ **L** is the mid-point of **BC**.

Note. The point **G** in which the three medians meet is called the **centroid** or the **mean-centre** of the triangle.

In Fig 310, it has been proved that **AG=GH**, and that **BGCH** is a parallelogram ; as the diagonals of a parallelogram bisect one another, **GH=2GL**.

∴ **AG=2 GL**.

∴ **GL**=$\frac{1}{3}$ **AL** and **AG**=$\frac{2}{3}$**AL**.

In a similar way it may be shown that **GM**=$\frac{1}{3}$ **BM** or $\frac{1}{2}$ **BG** and **GN**=$\frac{1}{3}$ **CN** or $\frac{1}{2}$ **CG**.

Thus **the centroid of a triangle is a point of trisection of each of the medians, the greater segment being towards the vertex.**

Ex. *If* **G** *is a point on the median* **AL** *such that* **AG=2GL**, *prove directly that* **BG** *produced bisects* **AC**, *and* **CG** *produced bisects* **AB**. [See Fig. 310].

IV. **The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.**

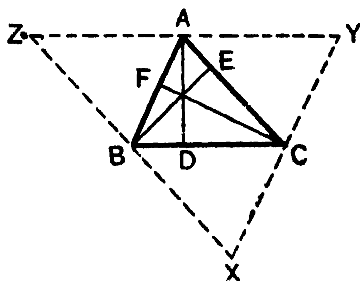


Fig. 311

ABC is a triangle. Let **AD**, **BE**, **CF** be the perpendiculars from **A**, **B**, **C** to **BC**, **CA**, **AB** respectively. To prove that **AD**, **BE**, **CF** are concurrent.

Through **A**, **B**, **C** draw parallels to **BC**, **CA**. **AB** respectively, intersecting one another in **X**, **Y**, **Z** as shown in Fig. 311.

Proof By construction **BCYA** is a ||m,

∴ **AY = BC**.

Since **BCAZ** is also a ||m, **ZA = BC**.

∴ **AY = ZA**,

that is, **A** is the mid-point of **YZ**.

Similarly it may be shown that **B** is the mid-point of **ZX**, and **C** that of **XY**.

Thus **A**, **B**, **C** are the mid-points of the sides **YZ**, **ZX**, and **XY** respectively of the △ **XYZ**.

Now since **YZ** is || to **BC**, and **AD** is ⊥ to **BC**.

∴ **AD** is ⊥ to **YZ**.

Similarly **BE** is ⊥ to **ZX** and **CF** is ⊥ to **XY**.

Thus **AD**, **BE**, **CF** are the perpendicular bisectors of the sides of the △ **XYZ**.

∴ **AD**, **BE**, **CF** are concurrent. (by I.)

Note. The point in which the perpendiculars drawn from the vertices of a triangle **ABC** to the opposite sides meet, is called the **orthocentre** of the triangle. This point is usually denoted by the letter **P**, just as the centroid is usually denoted by the letter **G**, and the circumcentre by the letter **O**.

MISCELLANEOUS EXERCISE I.

1. **ABCDEF** is a regular hexagon; prove that the triangle **BDF** is equilateral.

2. **OA**, **OB** are two radii of a circle at right angles to each other, and **CD** is any diameter. **AM**. **BN** are perpendiculars drawn to **CD**. Show that **OM = BN**, and **ON = AM**.

3. **ABC** is an isosceles triangle in which **AB = AC**. **BA** is produced to **D**, making **AD = AB**. Show that △ **BCD** is right-angled.

4. **ABC** is an equilateral triangle. On **BC, CA, AB** three other equilateral △s viz., **BCD, CAE, ABF** are described. Show that **EAF** is one straight line, so also **FBD** and **DCE**. Is the △ **DEF** equilateral ?

5. **AB, BC** are two consecutive sides of a regular polygon of $2n$ sides. Express ∠s **ABC** and **BAC**, (i) in right angles, (ii) in degrees.

6. If **BC** is the greatest side of a triangle **ABC**, and **P** is any point in **BC** (not produced), show that **BC** > **AP**.

7. **ABCDEF** is a regular hexagon show that the angle **ABD** is a right angle.

8. If the hypotenuse of a right-angled triangle be produced both ways, the sum of the exterior angles so formed is three right angles.

9. **ABC** is an equilateral triangle. **BC** is produced to any point **Q** ; show that **Q** is nearer to **A** than to **B**.

10. **ABC** is any triangle, and squares **ABDE** and **ACFG** are described on **AB, AC** external to the triangle. Show that **BG**=**CE**

11. **ABCD** is a parallelogram. In **AC** two points **E** and **F** are taken so that **AE**=**CF** ; show that **BEDF** is a parallelogram.

12. Prove that in every parallelogram one of the diagonals is greater than any of the sides.

13. From a point in the base of an isosceles triangle perpendiculars are drawn to the sides. Prove that the sum of these perpendiculars is the same for all positions of the point.

14. **ABCD** is a parallelogram. **CB** is produced to **E**, making **BE**=**AB**, and **CD** is produced to **F**, making **DF**=**AD** ; show that the points **E, A, F** are collinear.

15. **ABC** is a rignt-angled triangle with **BC** for hypotenuse. The bisector of ∠**C** cuts the perpendicular from **A** to **BC** in the point **D**, and the side **AB** in **E**. Show that △**ADE** is isosceles.

16. **ABCD** is a parallelogram. **BD** is produced to P making **DP**=**BD**. Through **P** a parallel to **DA** is drawn, cutting **BA** produced in **Q**. Show that **ACDQ** is a parallelogram.

17. **PQ** and **SR** are two diameters of a circle. Through **P** is drawn **PX**, ‖ to **RS**. Show that **PS**, **PR** are the bisectors, one internal, the other external, of the angle **QPX**.

18. **ABC** is any triangle in which ∠**B**=60°. Prove that **AC**+½**AB**>**BC**.

19. **PQ, a chord of a circle (centre O), is produced to R, such that QR=OP. RO is joined and produced to S, show that ∠POS= 3 ∠PRS.**

20. Points **P**, **Q**, **R**, **S**, are taken on the sides **AB**, **BC**, **CD**, **DA** of a parallelogram **ABCD**, such that **CR**=**AP**, and **DS**=**BQ**. Show that the figure **PQRS** is a parallelogram.

21. **ABCD** is a square, and **P** is any point on the diagonal **AC**. Show that the sum of the distances of **P** from **AB**, **BC** is the same for all positions of **P**.

22. **P**, **Q**, **R**, **S**, are four points such that **PQ**=**PR**, and **SQ**=**SR**; prove that any point on the straight line **PS** is equidistant from **Q** and **R**.

23. **ABCD** is a parallelogram, **O** is the mid-point of **AB**. If **OC**=**OD**, show that the parallelogram must be a rectangle.

24. **ABC** is an isosceles triangle, the base being **BC**; **P**, **Q** are points on **AB**, **AC** respectively such that ∠**ACP**=∠**ABQ**. Show that **PQ** is parallel to **BC**.

25. **ABC** is any triangle. Through the mid-point **D** of **BC**, a parallel to **AB** is drawn meeting the bisector of ∠**B**, in **P**; Show that ∠**BPC** is a right-angle.

26. **ABC** is a triangle; the bisectors of the ∠s **B**, **C** meet in **I**, and through **I** a parallel to **BC** is drawn meeting **AB**, **AC** in **P**, **Q** respectively. Show that **PQ**=**BP**+**CQ**.

27. **P, Q, R, S** are the mid-points of the sides **AB, BC, CD, DA** of a rectangle **ABCD**. Show that the figure **PQRS** is a rhombus.

28. If a pair of adjacent angles of a quadrilateral are together equal to the other pair of adjacent angles, the quadrilateral must have two of its sides parallel.

29. **ABCD** is a quadrilateral in which **BC = AD** and ∠**A** = ∠**B**, show that **AB** and **CD** are parallel.

30. **ABC** is a triangle, and two points, **X, Y** are taken on **AB, AC** respectively, such that **XY** is parallel to **BC**, and a fourth part of it. Show that **BX = 3AX**.

31. **ABCDE** is a regular pentagon, find (i) the angles of the △ **EBD**, (ii) the angles of the △ **ACO** where **O** is the point in which **AE** produced and **DC** produced meet.

32. **AD, BE**, are two medians of a triangle **ABC**, meeting in **G**; **L** and **M** are the mid-points of **AG, BG**. Show that **LMDE** is a parallelogram. Hence show that **G** is a point of trisection of **AD** and **BE**.

33. If two medians of a triangle are equal, the triangle is isosceles.

34. **ABC** is a triangle. Through **A**, a parallel to the median **BE** is drawn cutting **CB** produced in **K**. Show that **AK = 2BE**.

35 On a given straight line **BC** as base triangles are described. Through the vertex (opposite to **BC**) of each triangle a line is drawn parallel to the median from **B**. Show that all such lines are concurrent.

36. **ABC** is a triangle; **E, F** are the mid-points of the sides **AC, AB**, and the perpendicular drawn from **A** to **BC** meets **BC** in **D**. Show that ∠**EDF** = ∠**BAC**.

37. Prove that the sum of the medians of a triangle is greater than three quarters of the perimeter. [see Ex. 5 (iii), p. 143].

21

38. **ABC** is an isosceles triangle in which **AB=AC.** A straight line cuts **AB, BC** in **P, Q** respectively, and **AC** produced in **R.** If **PQ=QR** ; show that **BP=CR.**

39. **ABCDE,** and **ABPQRS** are respectively a regular pentagon and a regular hexagon described on opposite sides of the same line **AB.** If **PB** produced meets **CD** produced at **T,** prove that **CP=CT.** [$\angle$**CPT**$=24°=\angle$**CTP.**]

40. **ABCDEFGH** is a regular octagon, show that (i) the figures **ACEG, BDFH** are squares; (ii) the straight lines **AF, BE, CH, DG** form a square by their intersections, (iii) the inner figure common to the squares **ACEG, BDFH** is a regular octagon.

41. **ABCD** is a parallelogram whose diagonals meet in **E.** Any straight line is drawn through **A** so as not to cut the parallelogram, and perpendiculars **BM, CN, DP, EQ** are drawn to it. Show that **BM+CN+DP=4EQ.** [See Ex. 9, p. 228].

42. **ABC is a triangle having $\angle$C=90° and $\angle$A=30°. Show that, AB=2BC, and AC=$\sqrt{3}$ BC.**

43. The diagonals of a rhombus are 6 cm. and 10 cm. Calculate the length of a side.

44. A side of a rhombus is 8 cm. and a diagonal, 4 cm. Calculate the length of the other diagonal.

45. Draw two equal circles of radius 3 in. so that each passes through the centre of the other Calculate the length of the common chord to the nearest hundredth of an inch and verify by measurement. [Use a diagonal scale].

46. **ABC is a triangle, and a straight line is drawn parallel to BC cutting AB, AC in P,Q respectively. Show that PQ is bisected by the median AD.** [$\triangle$**ADB**$=\triangle$**ADC,** also $\triangle$**BPD**$=\triangle$**CQD,** $\therefore$ $\triangle$**APD**$=\triangle$**AQD** ; these being triangles of equal areas standing on opposite sides of a common side **AD,** **AD** bisects **PQ,** (see Ex. 1, p. 273)].

47. **ABCD** is a parallelogram. A straight line drawn parallel to **BC** cuts **AB**, **AC**, **CD** in **P**, **Q**, **R**; show that △s **ARD**, **ABQ** are equal in area.

48. Prove that if a diagonal of a quadrilateral bisects its area, it will bisect the other diagonal.

49. **ABC** is a triangle, and **D** is a point in **BC** such that $\frac{BD}{DC}=\frac{m}{n}$ where m and n are two integers, find the ratio of the area of the △**ABD** to that of the △**ABC**.

50. **ABC** is a triangle, with the sides **AB**, **BC**, respectively equal to 3 in. and 4 in.; **D** and **E** are points on **BC**, **BA** distant 2 inches from **B**. **AD** and **CE** are joined, and they meet in **F**. Show that **AF**=**FD** and **CF**=3 **FE** [△**EDC**=$\frac{1}{2}$ △**EBC**=△**EAC**, thus **EAC**, **EDC** are two △s equal in area standing on a common base **EC** and on opposite sides of it. **EC** bisects **AD**. [see Ex. 1, p. 273]

51. **ABC** is a triangle. **AB** is produced to **P** such that **BP**= **AB**; **PC** is produced to **Q**, such that **CQ**=**PC**; **QA** is produced to **R**, such that **AR**=**QA**. Show that the area of the △**PQR** is eight times the area of the △**ABC**.

52. Through two points taken on one of the diagonals of a parallelogram, lines are drawn parallel to the sides, dividing the parallelogram into nine parallelograms. Show that the two parallelograms remote from this diagonal are equal in area. [See § 123.]

53. **ABC** is a triangle From any point **P** on the side **BC**, parallels are drawn to **AB**, **AC**. Find the locus of the intersection of the diagonals of the parallelogram so formed.

54. **ABCD** is a fixed quadrilateral, find the locus of a movable point **P** such that **ABPD** is equal in area to **ABCD**.

55. In Ex. 54, find the locus of a point **P** such that area **ABPD** is half the area **ABCD**. [If **E** is the mid-point of **AC**, **ABED**=$\frac{1}{2}$ **ABCD**].

56. **AB** is a fixed straight line, and **O** a fixed point outside it **Q** is a movable point on **AB**; on **OQ** and on a specified side of it,

an equilateral triangle **OPQ** is described. Find the locus of **P**. If the triangle **OPQ** were described on the other side of **OQ**, what would be the locus of **P**? [See Ex. 18, (LIV)].

57. **O** is a fixed point, and **Q** any point on a fixed circle of centre **C**. **OQ** is joined and a line **OP** is drawn through **O** equal to **OQ** in length, such that ∠**QOP** is equal to a given angle in magnitude, and is in a specified sense. Show that the locus of **P** is a circle equal to the given fixed circle. [Proceed as in Ex. 18, (LIV)]

58. **O** is a fixed point, and **Q** any point on a fixed circle of centre **C** A square **OQRP** is described on **OQ**, on a specified side of it. Find the locus of **P**.

59. **O** is a fixed point and **Q** any point on a fixed circle of centre **C**. An equilateral triangle **OQP** is described on **OQ**, on a specified side of it. Find the locus of **P**.

60. Given the side **BC** of a triangle **ABC** in position, and the length of the median through **B**; show that the locus of **A** is a certain circle. [See Ex. 34].

61. Prove that the bisector of an angle **A** of a triangle **ABC** cannot meet the perpendicular bisector of the opposite side **BC** at a point within the triangle unless **AB**=**AC**. [Note that when **AB**=**AC**, the bisector of ∠**A**, and the perp. bisector of **BC** are coincident.]

62. In a triangle **ABC**, any two straight lines **BP**, **CQ** are drawn to meet **AC**, **AB** respectively in **P** and **Q**; show that **BP** and **CQ** can never bisect each other [For if **BP**, **CQ** bisected each other, the line joining the intersection of **BP**, **CQ** to the mid-point of **BC** would be parallel to both **AB**, **AC**; but this is absurd.]

63. **POQ** and **ROS** are two fixed intersecting lines. **X** is any point such that the sum of the perpendicular distances of **X** from the two lines is equal to a given length; show that the locus of **X** are the four sides of a certain rectangle [See Ex. 13; compare Ex. 9, (LIV).]

64. **POQ** and **ROS** are two fixed intersecting lines. Find the locus of a point such that its distance from **POQ** exceeds the distance from **ROS** by a given length. [Compare Ex. 10, (LIV)].

MISCELLANEOUS EXERCISE II.

1. With the aid of a ruler and a protractor construct a regular pentagon of side 1 inch.

2. Construct a regular hexagon of side 3 cm., (Use ruler and compasses only).

3. Construct a regular octagon of side 2 cm [You are not to use a protractor.]

4. Construct a convex quadrilateral **ABCD** in which **AB**=**BC** =1·2 in., ∠**B**=∠**ACD**=90°, **CD**=2**CA**. Make a triangle **CBE** equal in area to **ABCD**. **E** being a point on **BA** produced.

5. **ABC** is triangle having ∠**B**=50° and **BC**=2 in. Find the least possible length of **AC**, and the greatest possible value of ∠**C**.

6. Construct a parallelogram **ABCD** of area 24 sq. cm., and having **AB**=6 cm., and **AD**=5 cm

7. Devise as many ways as you can for trisecting a given line.

8. **PQRTU** is a field in the form of a pentagon. **Q** is 40 yd. south of **P** and 120 yd. **N. E.** of **R**; **T** is 200 yd. from **R** and 150 yd. from **Q**; **U** is 30° **E** of **N** from **P** and due east from **T**. Draw a plan of the field on a suitable scale and find its area.

9. Construct a triangle **ABC** having **AB**=2 in, **BC**=2·5 in., and **CA**=3 in Find a point in **BC** which shall be a quarter inch further from **AB** than from **AC**.

10. On a given straight line **AB** find a point **P** which shall be equidistant from two given points outside **AB**.

11. **Through a given point P to draw a straight line making equal angles with two given intersecting lines AOB, COD,** [There are two such lines, viz. the perpendiculars through **P** to the bisectors of the angles between **AOB**, **COD**.]

12. **A and B are two points on opposite sides of a straight line LM. Find a point P in LM, such that ∠APL=∠BPL.**

[Draw **AN** ⊥ to **LM** and produce it to **C**, such that **NC=AN**. Join **CB**, and let **CB** (or **BC**) produced meet **LM** in **P**; then **P** is the required point. Explain the case when **BC** is ∥ to **LM**.]

13. A and B are two points on the same side of a straight line LM. Find a point P in LM, such that ∠APL=∠BPM.

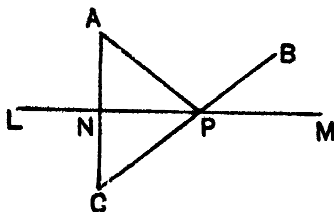


Fig. 312

[Draw **AN** ⊥ to **LM**, and produce it to **C** making **NC=AN**. Join **CB** and let it cut **LM** in **P**. Then **P** is the required point

∆s **APN, CPN** are congruent, ∴ ∠**APN**=∠**CPN**. But ∠**CPN**=vert. opp. ∠**BPM**.

∴ ∠s **APL, BPM** are equal.

14. A and B are two points on the same side of a straight line and P is any point in it. Show that AP+PB is least when P is such that ∠APL=∠BPM.

[Find the point **P** such that ∠**APL**=∠**BPM**, (see Ex. 13): take any other point **P′** in **LM**.

Since **LM** is the ⊥ bisector of **AC**, **AP′=P′C** and **AP=CP**.

∴ **AP+PB=CB**, and **AP′+P′B=CP′+P′B**.

But **CP′+P′B > BC**,

∴ **AP′+P′B > AP+PB**]

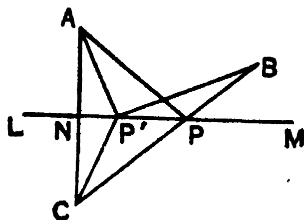


Fig. 313

The student will easily see that of all triangles with **AB** for a side, and the vertex opposite to **AB** being any point on **LM**, the triangle **APB** (Fig. 313) has the least perimeter.

15. LM is a fixed line, and A a fixed point outside it. P is a movable point on LM. AP is joined and a line PQ is drawn on the same side of LM as A, such that ∠QPM=∠APL. Show that QP produced passes through a certain fixed point for all positions of P.

16. **Show that of all triangles on a given base and of a given area the isosceles triangle has the least perimeter.** [See Ex. 14.]

17. **ABC** is any triangle. Describe a rhombus having two of its sides along **AB**, **AC**, and one vertex on **BC**.

18. **XOY is a given angle, and A is a given point within it. To draw through A a straight line such that the portion of it intercepted between OX, OY is bisected at A.** [Join **OA** and produce it to **L** making **AL = OA**. Through **L** draw a parallel to **OX** cutting **OY** in **M**. Join **MA** and produce it to cut **OX** in **N**, then **A** is the mid-point of **MN**. The proof is left to the student.]

19. **OP**, **OQ**, **OR** are three given straight lines, **OQ** lying between **OP** and **OR**. Through a given point **A** in **OP** to draw a straight line such that the portion of it intercepted between **OP** and **OR** is bisected by **OQ**. (Construct a parallelogram with **OA** for one side, another side lying along **OR**, and a diagonal lying along **OQ**).

20. **XOY is a given angle, and A is a point within it. Through A to draw a straight line cutting OX, OY in P and Q, such that the area of the △ OPQ is the least possible.**

[Through **A** draw a line cutting **OX**, **OY** in **P**, **Q** such that **AP = AQ** (see Ex. 18). Then △ **OPQ** is of the least possible area; for if any other line be drawn through **A** cutting **OX**, **OY** in **P′**, **Q′**, △**OP′Q′**, may be shown to be greater in area than △ **OPQ**. (Through **P** draw a parallel to **OY** cutting **P′Q′** in **Z**, and prove that △s **PAZ**, and **QAQ′** are congruent.)]

21. Show how to construct a rhombus **ABCD** so that the diagonal **AC** may be along a given straight line, and the sides **AB**, **BC**, **CD** may pass through three given points. [Use Ex. 12 or 13].

22. **XOY** is a given angle, and **A** a given point within it. Show how to draw through **A** a straight line, such that the portion of it intercepted between **OX** and **OY** may have the point **A** for a point of trisection.

[Join **OA** and produce it to **B** making $\mathbf{AB}=\frac{1}{2}\mathbf{OA}$. Through **B** draw a line cutting **OX, OY** in **P** and **Q** respectively such that $\mathbf{PB}=\mathbf{BQ}$ (see Ex, 18). By this construction **A** is the centroid of the $\triangle\mathbf{OPQ}$ (see § 135 III, Note) Let **PA** produced meet **OY** in **R** and **QA** produced meet **OX** in **S** Then **A** is a point of trisection of both **PR** and **SQ**. See Example, p. 317.]

23. Construct a triangle, having given the mid-points of the three sides. [See Ex. 1, p. 229].

24. Construct a triangle, given the lengths of two sides and of the median bisecting the third side. [Let a, b be the given lengths of the sides and k that of the median; construct a triangle **ACD** with $\mathbf{AC}=a$, $\mathbf{CD}=b$, and $\mathbf{AD}=2k$. Bisect **AD** at **L**. Join **CL** and produce it to meet the parallel to **CD** drawn through **A**, at **B**. Then **ABC** is the required triangle.]

25. Construct a triangle, given the lengths of one side and of the medians bisecting the other two sides [The data fix the triangle **BGC**, see Fig. 310.]

26. Construct a triangle, given the lengths of the three medians. [The data fix the $\triangle\mathbf{BGH}$, see Fig 310.]

27. Construct a triangle, given the mid-points of two sides, and the foot of the perpendicular drawn to the third side from the opposite vertex.

28. On a given straight line **AB** find a point (or points) which shall be equidistant from a given point **O** and another given straight line **CD**. [Plot the locus of points equidistant from **O** and **CD**, (Ex. 14, p. 304) and find the ponts where **AB** cuts this locus].

29. **AB** is a given straight line and **C, D** two given points. Find a point (or points) in **AB** such that the sum of the distances of the point from **C** and **D** may be equal to a given length. [Plot the locus of a point **P** such that $\mathbf{CP}+\mathbf{PD}=$given length (Ex. 1, p. 302), and find the points where **AB** cuts this locus].

30. **OX, OY, OZ** are three concurrent lines. **OY** lying within the angle **XOZ**, show how to draw through a given point **A** a straight line such that the portion of it intercepted between **OX,**

OY may be equal to the portion intercepted between **OY** and **OZ**. [Use Ex. 46, (Misc. Ex. I), and Ex. 19].

31. Construct a triangle, having given one of the sides, the difference of the angles at the extremities of this side, and the difference of the other two sides.

[Construct a Δ **BCD** such that **BC** is the given side of the required triangle, ∠**CBD**=half the given difference of the angles at the extremities of the given side, and **CD**=given difference of the other two sides (See Ex. 12. p, 85). From **B** draw a line **BE** on the same side of **BD** as **BC**, making ∠**EBD**=∠**CDB**; let **BE** and **DC** meet in **A**. Then **ABC** is the required triangle The proof is left to the student. When does the problem fail ?]

32. Construct a triangle having given one of the sides, the difference of the angles at the extremities of this side, and the sum of the other two sides. [Construct a Δ **BCD** such that **BC** is the given side of the required Δ, **CD**=given sum of the other two sides, and ∠**CBD**=complement of $\frac{1}{2}$ the given difference of the angles at the extremities of the given side. From **B** draw **BE** on the same side of **BD** as **BC**, making ∠**DBE**=∠**CDB**, and let **BE** cut **CD** in **A**. Then **ABC** is the required Δ. The proof is left to the student. Note that for a triangle to be possible it is necessary that **CD** should be greater than **BC**, so that ∠**CBD**>∠**CDB**.]

———:o:———

PART II.

THE CIRCLE.

CHAPTER XII.

PROPERTIES OF CIRCLES.

136. You are by this time quite familiar with the curved line to which the name '**circle**' has been given, and which is drawn with the aid of a pair of compasses; and you know what the following terms mean: **centre, radius, diameter, chord, arc, segment, sector, semi-circle**, etc.

You should re-read § 28—31, repeating Exercise VII (page 52).

It will not be out of place to recapitulate some of the essential points.

(i) A circle is a closed curve, it is the locus of all points which are at a specified distance (say a in.) from a certain fixed point, say O [see § 130]. Any point P, such that OP $<$ a in., *lies inside the circle*, and any point Q, such that OQ $>$ *a in., lies outside the circle.* Conversely if a point is inside the circle its distance from O is less than a in., and if it is outside the circle its distance from O is greater than *a in.* [See § 38]

(ii) Every diameter divides the circle into two halves; if the circle were folded about a diameter the two parts of the circle lying on opposite sides of it would

exactly fit : this shows that every diameter is an axis of symmetry of the circle (see § 30. and Note 4, § 57).

(iii) If the circumference of a circle is divided into a number of equal arcs, these arcs subtend equal angles at the centre. More generally, if two arcs of a circle are equal they subtend equal angles at the centre [see § 31].

Besides the above the following points may be noted :

(iv) A chord passing through the centre is a diameter.

(v) Every chord divides the circumference into two arcs. These arcs are equal if the chord passes through the centre, otherwise unequal ; in the latter case the greater arc is called a **major** arc and the lesser, a **minor** arc. Evidently a major arc is greater, and a minor arc is less, than half the circumference.

The major and minor arcs into which the circumference is divided by a chord are said to be **conjugate** to each other.

(vi) If two circles have equal radii, they are congruent [for if they are placed with their centres coincident, their circumferences will coincide]. Circles of equal radii will be called **equal circles**

We shall now proceed to study in a systematic and more scientific manner the fundamental properties of circles.

THEOREM 29. [Euclid III, 3].

137. (i) If a straight line drawn through the centre of a circle bisect a chord which does not pass through the centre, it cuts the chord at right angles.

Conversely, **(ii) the perpendicular to a chord from the centre bisects the chord.**

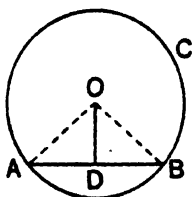


Fig. 314

(i) **ABC** is a circle whose centre is **O** : **AB** is a chord not passing through the centre, and **D** is its mid-point.

To prove that **OD** is ⊥ to **AB**.

Join **OA, OB.**

Proof. In the △s **OAD, OBD**

$$\begin{cases} OA = OB, & \text{(being radii)} \\ AD = BD, & \textit{(given)} \\ OD \text{ is common;} & \end{cases}$$

∴ the △s are congruent.

∴ ∠**ODA** = ∠**ODB**

∴ **OD** is ⊥ to **AB.**

(ii) **The converse Theorem. Let OD be ⊥ to AB.** (Fig. 314).

To prove that **AD = DB.**

Join **OA, OB.**

Proof. In the $\triangle^s$ OAD, OBD.

$\begin{cases} OA = OB, & \text{(being radii)} \\ OD \text{ is common,} & \\ \angle s\ ODA, ODB \text{ are rt. } \angle s; & (given) \end{cases}$

∴ the $\triangle s$ are congruent (*Theor.* 18)

∴ AD = DB.

Corollary 1. **The perpendicular bisector of a chord passes through the centre of the circle.** (See also Theor. 27)

Corollary 2. **A diameter of a circle bisects all chords drawn at right angles to it.**

[The diameter **POQ** bisects all chords **AB, A′B′,...** drawn at right angles to it [Fig. 315].

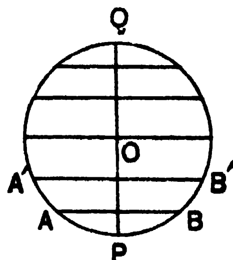


Fig. 315

Corollary 3. **The locus of the mid-points of a system of parallel chords is a diameter of the circle.** The proof is left to the student.

138. **Symmetry**—From Cor. 2 § 137, it follows that if the circle (Fig. 315) were folded about the diameter **PQ,** the point **B** would fall on the point **A, B′** on **A′** and so on, (for **PQ** bisects all these chords at right angles), hence the arc **PBQ** lying on one side of **PQ** would coincide with the arc **PAQ** lying on the other side of **PQ.** In other words, the circle is **symmetrical about the diameter PQ** (see Note 4, § 57).

An important property of symmetrical figures. Draw a line AB and mark any point P (Fig. 316). Draw PN ⊥ to AB and produce it to P′ such that NP′ = NP. It is evident that if the paper were folded about AB, the point P would come into coincidence with the point P′. The point P′ may be called the **image** of the point P with respect to the line AB.

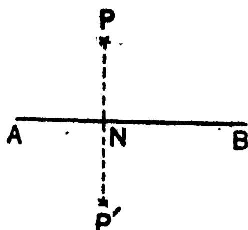


Fig. 316

Suppose AB is an axis of symmetry of the figure PQRR′Q′P′ (Fig. 317). Draw PL, QM, RN, ⊥ to AB, and let these lines when produced cut the

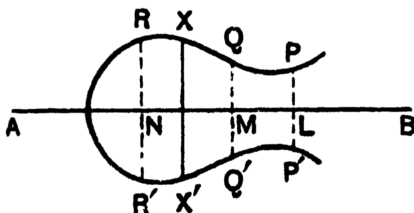


Fig. 317

figure again in P′, Q′, R′. If the figure is folded about AB, P will fall on a point in LP′, Q on a point in MQ′, and R on a point in NR′. Since AB is an axis of symmetry the part of the figure lying above AB will entirely coincide with the part below AB when the figure is folded about AB. Hence P must fall on P′, Q on Q′ and R on R′. Therefore PP′, QQ′, RR′ are bisected by AB, the axis of symmetry. Indeed if any

line is drawn ⊥ to **AB**, cutting the figure in two points **X** and **X′**, then **XX′** will be bisected by **AB**. **P′**, **Q′**, **R′**, **X′** are images of **P**, **Q**, **R**, **X** respectively. Thus *if a figure is symmetrical about a certain line, and* **P** *is any point on the figure, then the image of* **P** *will also be a point on the figure,* and we may say *that the part of the figure on one side of the axis of symmetry is an image of the part lying on the other side of it.*[1]

Conversely if a line exists such that the part of a certain figure lying on one side of the line is the image of the part lying on the other side of it, the line is an axis of symmetry of the figure.

139. **Symmetry about a point.** You can see that a rhombus is symmetrical about its diagonals ; a square has four axes of symmetry, namely, the two diagonals and the two lines bisecting the pairs of opposite sides ; a rectangle (which is not a square) has two axes of symmetry namely the two lines bisecting the pairs of opposite sides. But a parallelogram which is neither a rectangle nor a rhombus, has no axis of symmetry.

Let us take such a parallelogram **ABCD** (Fig. 318). **O** being the intersection of the diagonals. The diagonals **AC, BD** are bisected at **O** ; indeed it is easily shown that any line **PQ** drawn through **O** and terminated by the sides of the parallelogram has **O** for its mid-point.

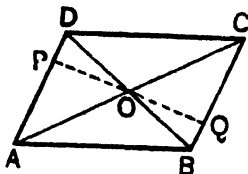


Fig. 318

1. By folding a piece of paper and cutting it into any shape, and then opening out the paper, you get a figure which is symmetrical about the crease.

[You can prove that $OP = OQ$, by showing that the $\triangle^s$ **OPD, OQB** are congruent]. We say that *the parallelogram* **ABCD** *is symmetrical about the point* **O**, though it has no axis of symmetry. The point **O** may be called the **centre of symmetry.**

The test of symmetry of a figure about a point **O** is that if any point **P** on the figure be joined to **O**, and **PO** produced to **P′** such that $OP' = PO$, then **P′** should also be a point on the figure.

Any parallelogram is symmetrical about the intersection of its diagonals. The centre of a circle is a centre of symmetry.

It may be noted that a figure may have a centre of symmetry without having an axis of symmetry, or it may have an axis of symmetry without having a centre of symmetry. Fig. 319 is an example of a figure which has an axis of symmetry but no centre of symmetry. Fig. 320 is an example of a figure which has a centre of

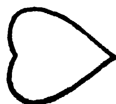


Fig. 319



Fig. 320

symmetry **O** but no axis of symmetry. A circle is an example of a figure which has axes of symmetry as well as a centre of symmetry.

EXERCISE LV.

1. A complete circle being given in position, to find its centre.

[Draw any chord **AB**, and let the perpendicular bisector of **AB** meet the circle in **C** and **D**. Then the mid-point (**O**) of **CD** is the required centre of the circle.]

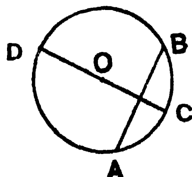


Fig. 321

2. An arc of a circle being given in position, to find its centre.]

[Draw two chords **AB, CD** so that they are not parallel. The intersection **O** of the perpendicular bisectors of **AB** and **CD** is the centre of the circle (see Cor. 1, Theor. 29). This method may also be used when the whole circle is given in position (Ex. 1)]

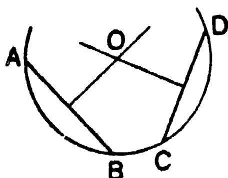


Fig. 322

3. Place a pice on the paper and trace an arc of a circle (not the complete circle) by moving a pencil point round it. Remove the pice and complete the circle.

4. The locus of the centres of circles passing through two given points A, B, is the perp. bisector of AB.

5. Through a given point within a circle draw a chord which shall be bisected at that point.

6. Show that two chords of a circle cannot bisect each other unless they are both diameters of the circle.

7. The line joining the mid-points of any two parallel chords of a circle passes through the centre.

8. Draw a circle of any radius and place any chord **AB** in it. Draw five other chords all parallel to **AB**. Bisect all these chords and verify that the mid-points lie on a straight line passing through the centre.

9 **AB, AC** are the equal sides of an isosceles triangle **ABC.** A circle with centre **A** and any suitable radius cuts **BC** (or **BC** produced) in **D** and **E** Show that **BD = CE**.

10. If a straight line cuts two concentric circles, show that the portions intercepted between the circumferences are equal.

11. From any point **O** two equal straight lines **OP**, **OQ** are drawn to a circle; show that the bisector of the angle **POQ** passes through the centre of the circle.

12. In a circle of diameter 10 feet, a chord 8 feet in length is placed. How far is it from the centre? [Use Theor. 25].

13. In a circle of diameter 6 inches, a chord is drawn at a distance of 2 inches from the centre. What is the length of the chord?

14. In a circle of radius 5 cm. two parallel chords are of lengths 8 cm. and 6 cm. What is the distance between the chords, (i) when the centre is on the same side of the chords, (ii) when the centre lies between the chords?

15. In a circle of diameter 200 yards, the distance between two parallel chords is 100 yards, and the centre lies between them. If one of the chords is 120 yards long, what is the length of the other?

16. **AB**, **AC** are two chords at right angles, of a circle of radius 120 ft. If **AB**=180 ft., find the length of **AC**, and also the distance between the mid-points of **AB** and **AC**.

17. A chord of a circle is 4 ft. in length and 10 ft. from the centre. Calculate the length of the radius.

18. How many axes of symmetry are there in a circle?

19. Draw a semi-circle on a given line **AB** as diameter, and construct an axis of symmetry. How many axes of symmetry are there? Has the arc of the semi-circle any centre of symmetry?

20. Describe the symmetry of the following figures:

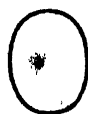
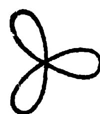

Fig. 323


Fig. 324


Fig. 325


Fig. 326


Fig. 327

21. Draw a curved line of any shape and a straight line **AB** of indefinite length. Construct the image of the curved line with respect to **AB**.

22. Prove that the line joining the centres is an axis of symmetry of two intersecting circles.

23. Two circles intersect in **A**, **B**; show that **AB** is bisected by the line joining the centres.

24. Two circles intersect in **A** and **B**. A parallel to **AB** cuts the circles in **P**, **Q**, **R**, **S**, the points being *in order*. Show that **PQ**=**RS**.

THEOREM 30.

140. There is one circle, and only one, which passes through three given points not in a straight line.

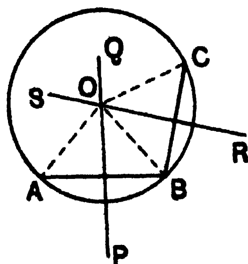


Fig. 328

A, B, C, are three points not in a straight line.

To prove that one circle, and only one, can pass through the points **A, B, C.**

It is only necessary to show that there is a point and only one such point which is equidistant from A, B, C.

Proof. The locus of all points equidistant from A and B is PQ, the perpendicular bisector of AB;

so also, the locus of all points equidistant from B and C is RS, the perpendicular bisector of BC.

Since A, B, C are not in one straight line, PQ and RS cannot be parallel and will intersect, and that in one point only.[1]

Let PQ and RS intersect in O. The point O is equidistant from A and B, and also from B and C;

∴ O is equidistant from A, B, C; and there is no other point equidistant from A, B, C.

Hence a circle with centre O and radius OA will pass through A, B, C and there is no other circle passing through A, B, C.

Cor. 1. **If the positions of three points on a circle are known, the size and position of the circle are completely determined.**

Cor. 2. **Two circles cannot intersect in more than two points.** For if the two circles have three points in common, they have the same centre and radius, and therefore coincide.

Cor. 3. **If two circles have a common arc, they must be one and the same circle.**

Note. A, B, C, D are four given points no three of which are in a straight line. Can a circle be described which shall pass through all the four points? There is a unique circle which passes through three of the points, say A, B, C. This circle may or may not pass

1. For two *straight* lines cannot intersect in more than one point. See also cor., Theor. 13 B.

through **D**. In case it passes through **D**, all the four points **A, B, C, D** lie on a circle.

If a number of points are such that a circle passes through them all, they are said to be **concyclic.**

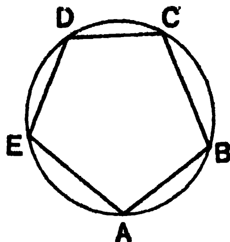


Fig. 329

If a polygon is such that a circle can be drawn passing through all the vertices of the polygon, the circle is said to be **circumscribed** about the polygon; and the polygon is said to be **inscribed** in the circle (Fig. 329).

Corollary 4. **One circle, and only one, can be circumscribed about any triangle ABC.**

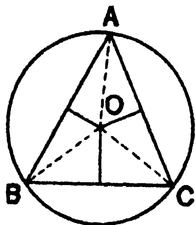


Fig. 330

To circumscribe a circle about a given triangle (Euclid IV. 5.) is the same problem as that of drawing a circle so as to pass through three given points not in a straight line, viz., the vertices of the Δ. The circle circumscribed about a triangle is called the **circum-circle,** its centre, the **circum-centre,** and its radius, the **circum-radius.** Evidently the circum-centre is the point **O** where the perpendicular bisectors of two sides of the triangle meet: the perpendicular bisector of the third side also passes through this point. [see § 135, I].

EXERCISE LVI.

1. How would the proof of Theorem 30 fail, if the points **A, B, C** were in a straight line?

2. How many circles can be circumscribed about a given triangle?

3. Construct a triangle **ABC**, having **BC**=6 cm., **AC**=5 cm. and **AB**=4 cm. Draw the circum-circle of the triangle and measure the radius.

4. Draw a circle passing through a given point and having a given line for a chord.

5. **Draw an acute-angled triangle, and construct the circum-circle. Does the centre lie inside the triangle? Can it lie outside the triangle?**

6. **Draw a right-angled triangle, and construct the circum-circle. Does the centre lie on the hypotenuse?**

7. **Draw an obtuse-angled triangle, and construct the circum-circle. Does the centre lie outside the triangle? Can it lie inside the triangle?**

8. Draw a large-sized triangle **ABC**. Mark the mid-points **L, M, N** of **BC, CA, AB**. Draw the circum-circles of the Δs **ABC, LMN**. Measure the radii and calculate their ratio.

9. Draw a large-sized acute-angled triangle **ABC**. From **A, B, C** draw perpendiculars **AD, BE, CF** to the opposite sides. Draw the circum-circles of the Δs **ABC, DEF**. Measure the radii and calcutate their ratio.

10. Draw the circum-circles of the two Δs ABC_1 and ABC_2 obtained in Ex. 12 p. 85, and measure the radii.

11. Draw a large-sized scalene triangle **ABC**; on the sides of this triangle draw equilateral triangles external to the Δ **ABC**. Draw as accurately as you can the circum-circles of these equilateral triangles. Do you find the circles pass through a common point?

12. Show that the circum-centre of an equilateral triangle is equidistant from the sides.

13. Show that the circum-centre of a right-angled triangle is the mid-point of the hypotenuse.

14. **ABCD** is any rectangle. Prove that a circle can be circumscribed about it [The vertices are equidistant from the intersection of the diagonals].

15. **If a point O within a circle is equidistant from three points on the circle, show that O must be the centre of the circle.** [Euclid III, 9].

16. **A** and **B** are two given points. Draw a circle which will pass through **A** and **B**. How many such circles can you draw?

17. Take two points **A** and **B**, 2·5 inches apart. Draw a circle of radius of 1·5 in. passing through **A** and **B**. How many such circles are possible? [Use Ex. 4, p. 338].

18. In Ex. 17, try to draw a circle of radius 1 inch passing through **A** and **B**. Can you draw such a circle? If not, why?

19. Construct a circle passing through two given points and having its centre on a given straight line. How many such circles are possible? When does the construction fail?

THEOREM 31. [Euclid III, 14].

141. (i) Equal chords in a circle are equidistant from the centre.

Conversely, **(ii) chords which are equidistant from the centre are equal.**

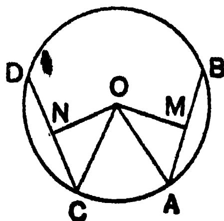


Fig. 331

(i) **AB, CD** are equal chords of a circle with centre **O**, and **OM, ON** are perpendiculars drawn from **O** to **AB, CD** respectively.

To prove that **OM = ON.**

Join **OA, OC.**

Proof: Since **OM** is ⊥ to **AB, M** is the mid-point of **AB.** (*Theor.* 29)

∴ **AM** = ½ **AB.**

Similarly **CN** = ½ **CD.**

But **AB = CD.** (*Given*)

∴ **AM = CN.**

Now in the right-angled △s **OAM, OCN,**

{hypotenuse **OA** = hypotenuse **OC,**
{**AM = CN,** (*Proved*)

∴ the △s are congruent, (*Theor.* 18)

∴ **OM = ON.**

(ii) **AB, CD** are two chords such that the perpendiculars **OM, ON** drawn from the centre **O** to **AB, CD,** are equal. (Fig. 231).

To prove that **AB = CD.**

Join **OA, OC.**

Proof. As in (i), it may be shown that

AB = 2AM

and **CD = 2CN**

Now in the right-angled △s **OAM, OCN,**

{hypotenuse **OA** = hypotenuse **OC,**
{**OM = ON,** (*Given*)

∴ the △s are congruent, (*Theor,* 18)

∴ **AM = CN,**

∴ **2 AM = 2 CN,**

that is, **AB = CD.**

Note. The above theorems may also be established in the following way.

Let **AB** be a chord of a circle with centre **O**, and let **M** be its mid-point. Join **OA**, **OM**. By Theorem 29, **OM** is ⊥ to **AB**.

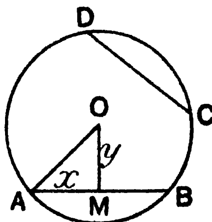


Fig. 332

By the Theorem of Pythagoras,

$$\mathbf{OA^2 = AM^2 + OM^2}.$$

If the radius of the circle be a units of length, the semi-chord $\mathbf{AM}=x$ units of length, and the distance of the chord from the centre i. e. $\mathbf{OM}=y$ units of length, we have,

$$a^2 = x^2 + y^2 \qquad \text{(i)}$$

Let us take any other chord **CD**: if half this chord be x' units of length and its distance from the centre be y' units of length, we shall have in exactly the same way.

$$a^2 = x'^2 + y'^2. \qquad \text{(ii)}$$

From (i) and (ii) we have,

$$x^2 + y^2 = x'^2 + y'^2 \qquad \text{(iii)}$$

The following inferences from (iii) should be noticed.

(1) If $x'=x$, then $y'=y$;

Thus *if two chords* **AB, CD** *are equal, they are equidistant from the centre.*

(2) If $y'=y$, then $x'=x$;

Thus *if two chords* **AB, CD** *are equidistant from the centre, they are equal.*

(3) If $x'>x$, then $y'<y$;

Thus *if two chords* **AB, CD** *are unequal the greater chord is nearer to the centre.* [Euclid III. 15.]

(4) If $y'>y$, then $x'<x$,

Thus *if two chords* **AB, CD** *are at unequal distances from the centre, then that one of the chords which is nearer to the centre is the greater.*

From the relation (i) you can calculate the length of a chord when its distance from the centre is known, or the distance of the chord from the centre when its length is known [see note 2, § 127]. Repeat Exs. 12—16. Exercise LV.

EXERCISE LVII.

1. **In a given circle to place a chord equal to a given line.** [Euclid IV. 1].

[With any point **A** on the circle as centre, and radius equal to the given line, draw an arc of a circle cutting the given circle in **B**. Then **AB** will be a required chord. When does the problem fail ?]

2. Draw a circle of radius 1·5 inches, and place two chords in it each of length 2 inches. Measure the distances of the two chords from the centre of the circle. Do you find the distances equal ?

3. Draw a circle with any point **O** as centre and of radius 1·5 inches ; mark a point **A** in it and through **A** draw two chords **AB**, **AC**, one of length 2 inches, the other of length 2·5 inches. Find by measurement the distances of these chords from **O**. Verify the results by calculation.

4. From any point **P** within a circle with centre **O**, two chords are drawn which are equally inclined to **PO**. Prove that the chords are equal.

5. **AB** is a chord of a circle, and **P** a point in it (not the mid-point). Through **P** draw another chord of the same length as **AB**.

6. **AB** is a chord of a circle and **P** is a point in **BA** produced. Through **P** draw a straight line cutting the circle in points **R** and **S** such that **RS**=**AB**.

7. **AB** and **CD** are two equal and parallel chords of a circle. Show that **A**, **B**, **C**, **D** are the vertices of a rectangle.

8. **AB** is a chord of a circle. Through **A**, **B** two other chords **AC**, **BD** are drawn on the same side of **AB**, such that ∠**CAB**= ∠**DBA**. Prove that **AC**=**BD**.

9. **Show that the locus of the mid-points of equal chords of a circle is a concentric circle.**

10. Place any chord **AB** in a circle and draw the locus of the mid-points of all chords equal to **AB**.

11. Two equal chords of a circle intersect in a point; show that the segments of the one are equal respectively to the segments of the other.

12. Show that a diameter is the greatest chord that can be drawn in a circle. [Let O be the centre and **AB** a chord not passing through the centre, consider the Δ**AOB** and use Theor. 11]

13. **P** is a point within a circle, distinct from the centre. Draw the chord through **P** which shall have **P** for its mid-point; and show that it is the shortest of all chords that can be drawn through **P**. What is the longest chord through **P**?

14. In a given circle draw a chord of given length and parallel to a given line. [Proceed by placing *one* chord of the given length (Ex. 1)]

15 Through a given point **P** within a circle draw a chord of given length. When does the problem fail?

16. Calculate the length of the shortest chord of a circle of radius 10 feet that can be drawn through a point distant 8 feet from the centre.

17. Construct a chord of a circle equal in length to a given chord and making a given angle with it. [See Ex. 13, p. 200].

18. **P** *is a point within a circle, distinct from the centre* **O** (Fig. 333), *and* **AB** *is the diameter through* **P**. *To prove that* **PA** *is the least and* **PB**, *the greatest of all straight lines that may be drawn from* **P** *to the circumference.* [Euclid III 7.]

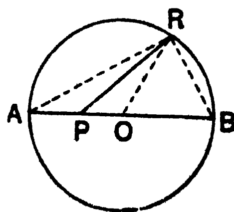


Fig. 333

[Let any line **PR** be drawn from **P** to the circumference. Join **OR, RA, RB.**

Now **PR**+**PO** > **OR** (Theor. 11); but **OR**=**OA**.

PR+**PO** > **OA**, ∴ **PR** > **PA**.

Again **PO**+**OR** > **RP** (Theor. 11), but **OR**=**OB**.

PO+**OB** > **RP**

i. e. **PB** > **RP**]

19. ***P** is any point outside a circle with centre* **O**, *and* **PAB** *is the straight line through* **P** *and* **O** *cutting the circle in* **A** *and* **B** (*Fig.* 334). *Prove that* **PA** *is the least and* **PB** *the greatest of all straight lines that may be drawn from* **P** *to the circumference.* [Euclid III, 8.]

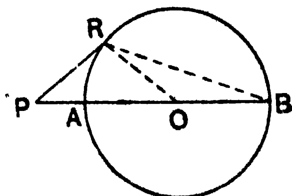


Fig. 334

EXERCISE LVIII.

1 Every circle which passes through a fixed point **A**, and has its centre on a given straight line **PQ**, passes through another fixed point. [The other fixed point is the image of **A** with respect to **PQ**, see § 138.]

2. Two circles intersect in **P** and **Q**; a line is drawn through **P** parallel to the line of centres and terminated by the circles. Prove that this line is double the distance between the centres.

3. Two circles intersect in **P** and **Q**. Two parallel straight lines **APB**, **CQD** are drawn through **P** and **Q** cutting the circles again in **A**, **B**, and **C**, **D**, respectively. Show that **AB**= **CD**.

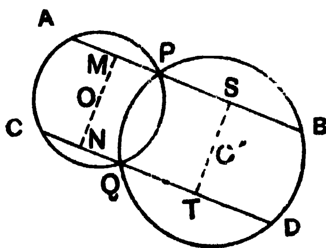


Fig. 335

[Draw perpendiculars **MN**, **ST** to **AB**, **CD** through the centres. Then **MSTN** is a rectangle, and **MS**=**NT**. Hence prove that **AB**=**CD**].

4. Two circles intersect, and through a point of intersection **P**, two straight lines **APB** and **CPD** are drawn which are equally inclined to the common chord, and cut the circles again in **A**, **B**, and **C**, **D**. Show that **AB**=**CD**.

5 Two circles whose centres are **A**, **B** intersect in **P** and **Q**, **O** is the mid-point of **AB**, and a straight line **CPD** is drawn through **P** perpendicular to **OP**, cutting the circles again in **C**, **D**. Prove that **CP**=**PD**.

6. Through a point of intersection of two circles draw a straight line such that the chords intercepted on it by the two circles are equal. [Use Ex. 5].

7. Two circles of radii 50 cm., and 80 cm., have their common chord=60 cm. Calculate the distance between the centres. Draw a diagram on a scale of 1 cm. to 10 cms. and verify your result by measurement.

8. Two circles of radii 50 cm. and 80 cm. have their centres 100 cm. apart Calculate the length of the common chord. Verify your result by drawing a figure on a scale of 1 cm. to 10 cms. [See § 129, II].

9. A circle cuts two concentric circles in **P**, **P′** and **Q**, **Q′** respectively; show that **PQ**=**P′Q′**. [Consider the symmetry of the figure about the line of centres].

ANGLES IN SEGMENTS.

THEOREM 32. [Euclid III, 20.]

142. The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining portion of the circumference.

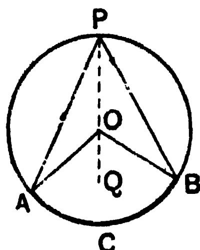


Fig. 336

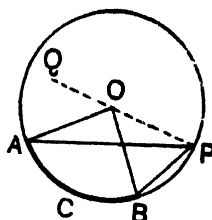


Fig 337

The arc ACB subtends ∠AOB at the centre O, and ∠APB at P, any point on the remaining part of the circumference.

To prove that ∠AOB = 2 ∠APB.

Join PO and produce it to Q.

Proof. Case 1. When the centre O is within ∠APB. [Fig. 336].

In △ OAP, OA = OP (being radii),

∴ ∠OAP = ∠OPA.

Now ∠AOQ is an exterior ∠ of △ OAP.

∴ ∠AOQ = ∠OAP + ∠OPA, [Theor. 16, Corollary.]

= 2 ∠OPA.

1 Euclid enunciates the Theorem in the following way. *'In a* *rcle the angle at the centre is double of the angle at the circumference,* *en the angles have the same circumference as base'.*

Similarly, $\angle BOQ = 2\ \angle OPB$:

$\therefore\ \angle AOQ + \angle BOQ = 2\,(\angle OPA + \angle OPB)$,

that is, $\angle AOB = 2\ \angle APB$.

Case II. When the centre O is outside $\angle APB$ [Fig. 337]

As before, $\angle QOB = 2\ \angle OPB$.

and $\angle QOA = 2\ \angle OPA$.

$\therefore\ \angle QOB - \angle QOA = 2\,(\angle OPB - \angle OPA)$

that is, $\angle AOB = 2\ \angle APB$.

Prove the case when **AP** or **BP** passes through **O**.

Note. If an arc **ACB** is a semi-circle the angle **AOB** which it subtends at the centre O of the circle is two right angles [Fig. 338].

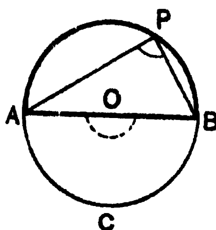


Fig. 338

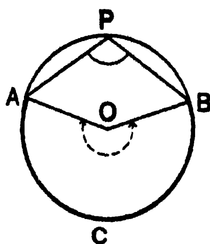


Fig. 339

If an arc **ACB** be less than a semi-circle, *the angle it subtends at the centre is less than 2 right-angles* (Fig. 336): and if the arc **ACB** is greater than a semi-circle, *the angle it subtends at the centre is greater than 2 right-angles* but less than 4 right angles (Fig. 339). In each case the angle **APB** subtended at a point **P** on the remaining arc is $\frac{1}{2}$ $\angle$**AOB**.

143. A segment of a circle is a figure bounded by an arc and the *corresponding* chord [see § 29], the arc and the chord being called the arc and the chord of the segment.

A segment of a circle is called a **major segment or a minor** segment according as it is greater or less than a semi-circle.

By **angles in a segment** of a circle we mean the angles which the chord of the segment subtends at different points on its arc. Thus ∠s APB, AQB, are angles in the segment APB (Fig. 340). It will be proved in the next theorem that all the angles in the same segment of a circle are equal.

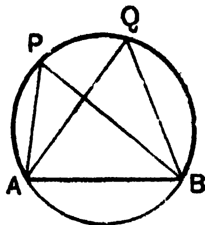


Fig. 340

THEOREM 33. [Euclid III, 21].

Angles in the same segment of a circle are equal.

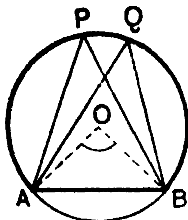


Fig. 341

APB, AQB are two angles in the same segment APQB.

To prove that ∠AQB = ∠APB.

Join A, B to the centre O of the ⊙.

Proof. $\angle APB = \frac{1}{2} \angle AOB$ [*Theor.* 32]

also $\angle AQB = \frac{1}{2} \angle AOB$.

∴ $\angle AQB = \angle APB$.

Note. In the above (Fig. 341) the segment **APQB** considered is a major segment. The theorem holds for all segments, and

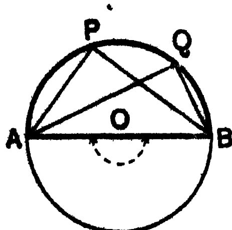


Fig. 342

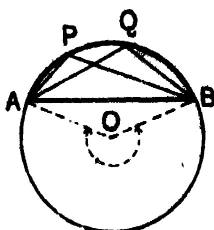


Fig. 343

the proof is the same for all cases. Figs. 342, 343 represent the cases, when the segment (**APQB**) is (i) a semi-circle. (ii) a minor segment.

Since all the angles in a segment of a circle are equal, we shall speak of **the angle in a segment** as the angle of definite magnitude to which all the angles in the segment are equal.

If two segments of circles contain equal angles they are said to be similar.

Corollary. **If two segments of circles contain equal angles and their chords are equal, they are congruent, and are segments of equal circles.** [For if the segments were placed with their chords coincident, their arcs would also coincide ; for if not, by drawing a line **APQ** through an extremity **A** of the common chord **AB**, cutting the arcs in **P** and **Q**, it is easily shown that ∠s **APB**, **AQB** are unequal (see Theor. 8, § 65; see also Cor. 3, § 140)].

THEOREM 34. [Euclid III, 31].

144. (i) **The angle in a semi-circle is a right angle,** (ii) **the angle in a major segment is an acute angle and** (iii) **the angle in a minor segment is an obtuse angle.**

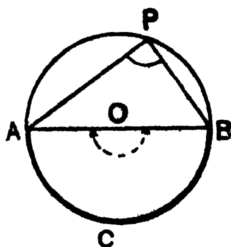


Fig. 344

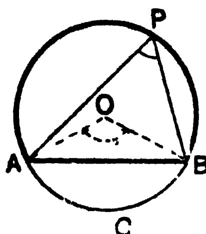


Fig. 345

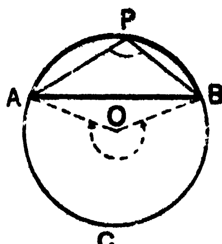


Fig. 346

(i) **APB** is a semi-circle (Fig. 344) ; *to prove that* ∠**APB** = a right angle.

Proof : Since **APB** is a semi-circle,

∴ **ACB** is also a semi-circle,

∴ ∠**AOB** is two right-angles.

But ∠**APB** = $\frac{1}{2}$ ∠**AOB**. (*Theor.* 32)

∴ ∠**APB** = a right-angle[1].

(ii) **APB** is a major segment (Fig. 345) ; *to prove that* ∠**APB** is an acute angle.

Proof : Since **APB** is a major arc,

∴ **ACB** is a minor arc, [see § 136, (iv)]

∴ ∠**AOB** is less than two right-angles.

But ∠**APB** = $\frac{1}{2}$ ∠**AOB**.

∴ ∠**APB** is less than a right angle, and is, thus an acute angle.

---

1 For another proof see Note, § 105.

(iii) **APB** is a minor segment (Fig. 346); *to prove that* $\angle$**APB** is an obtuse angle.

Proof. Since **APB** is a minor arc,

$\therefore$ **ACB** is a major arc.

$\therefore$ $\angle$**AOB** is greater than 2 right angles but less than 4 right angles.

But $\angle$ **APB** $= \frac{1}{2}$ $\angle$**AOB**,

$\therefore$ $\angle$**APB** is greater than 1 right angle but less than 2 right angles, and is thus an obtuse angle.

Note. The converse of Theorem 34 is also true. *If the angle in a segment be a right angle, the segment must be a semi-circle, if the angle be acute, the segment must be a major segment, and if the angle be obtuse, the segment must be a minor segment.*

EXERCISE LIX.

1. **P, Q, R** are three points on the circumference of a circle whose centre **O** lies within $\angle$**QPR**.
   (i) If $\angle$**OQR**$=20°$, find $\angle$**QPR**.
   (ii) If $\angle$**OQR**$=30°$ and $\angle$**OPR**$=15°$, find $\angle$s **PRQ, POQ, PQR; RPQ**.
   (iii) If $\angle$**PRQ**$=60°$, **RQO**$=50°$, find $\angle$s **QPR** and **PQO**
   (iv) If $\angle$**POR**$=110°$, $\angle$**POQ**$=150°$, find $\angle$**QPR**.

2. In Ex. 1. (iv) find $\angle$**QPR** if **O** does not lie within the $\angle$**QPR**.

3. Are $\angle$s **PAQ, PBQ** equal in Fig. 340? Give reasons for your answer.

4. **A, B, C** are three points on a circle whose centre is **O.** If $\angle$**AOB**$=110°$, $\angle$**BOC**$=138°$, find the angles of the triangle **ABC.**

5. In Fig. 345, show that $\angle$**APB**$+\angle$**OAB**$=1$ right angle.

6. In Fig. 346, show that $\angle$**APB**$-\angle$**OAB**$=1$ right angle.

7. A segment of a circle has its chord = 5 cm. and height[1] = 3 cm. Find the angle of the segment. [Take **AB** = 5 cm.; bisect it at **M**, and through **M** erect a perpendicular **MO** to **AB**, and cut off **MO** = 3 cm. Join **AO**, **BO**. Then ∠**BOA** is an angle of the segment. Measure it with a protractor.]

8 A segment of a circle has its chord = 100 feet, and height = 40 feet. Find the angle of the segment [Draw a figure as in Ex. 7, on a suitable scale].

9. **P** is any point on the arc of a segment of which **AB** is the chord. Show that ∠**PAB** + ∠**PBA** is constant.

10. A circle is circumscribed about an equilateral triangle. Find the angle in each of the segments which lie outside the triangle.

11. Two circles intersect in **A** and **B** Through **A**, diameters **AP** and **AQ** are drawn (one for each circle), show that **P**, **B**, **Q** are in one straight line.

12 **P**, **Q**, **R**, **S** are four points on a circle being the vertices of a quadrilateral **PQRS**. The diagonals **PR**, and **QS** meet in **X**. Show that △s **PXS** and **QXR** are equiangular[2], as also the △s **SXR**, and **PXQ**.

13. Through a point **X** within a circle with centre **O**, two chords **PR**, **QS** are drawn. Show that ∠**POQ** + ∠**ROS** = 2∠**PXQ**.

14 Two chords **PR**, **QS**, of a circle (centre **O**), intersect at a point **X** when *produced in the directions* **PR**, **QS**, show that ∠**POQ** - ∠**ROS** = 2∠**PXQ**

15. Draw three radii **OA**, **OB**, **OC** of a circle so that ∠**AOB** = ∠**BOC** = 120° Show that △**ABC** is an equilateral triangle.

---

1. The height of a segment is the length of the perpendicular drawn to the chord from its mid-point and terminated by the arc and the chord of the segment.

2. When we say "a triangle is equiangular" we mean that its angles are equal to one another.

When we say "two triangles are equiangular" we mean that the angles of the one are equal respectively to the angles of the other.

16. **PQ**, **RS** are two diameters of a circle at right angles to each other. Show that **PSQR** is a square

17. In Ex. 15 draw radii **OE**, **OF**, **OG** bisecting the $\angle$s **AOB**, **BOC**, **COA** respectively. Show that the figure **AEBFCG** is a regular hexagon.

18. Two circles **C** and **C′** intersect in **A** and **B** Two straight lines **PAQ**, and **LAM** are drawn through **A** cutting the circles **C**, **C′** in **P**, **Q** and **L**, **M** respectively. Show that $\angle$**PBL** = $\angle$**QBM**. Show also that the $\triangle$s **PBQ**, and **LBM** are equiangular.

19. **ABC** is a triangle; **AD** is the perpendicular from **A** to **BC**, and **AE** is the diameter of the circum-circle, through **A** Prove that the $\triangle$s **ABD**, **AEC** are equiangular, so also the $\angle$s **ACD**, **AEB**.

20. The bisector of $\angle$**A** of a $\triangle$**ABC** cuts **BC** in **D** and the circum-circle in **E**. Show that the triangles **ABD** and **AEC** are equiangular.

21. Prove that if the internal and external bisectors of an angle of a triangle cut the circum-circle again in the points **P** and **Q**, then **PQ** is a diameter of the circum-circle

22. The circle described on one of the equal sides of an isosceles triangle as diameter, bisects the base.

23. The circles described on two sides of a triangle as diameters, intersect again on the third side. [Cf. Ex. 11].

24. The four circles described on the sides of a rhombus as diameters pass through a common point.

25. **ABC** is an equilateral triangle inscribed in a circle In the minor arc **AB** any point **P** is taken. From **PC** a part **PX** is cut off equal to **PB**. Show that (i) the $\triangle$ **PBX** is equilateral, (ii) the $\triangle$s **APB** and **BCX** are equiangular, and (iii) **PC** = **PA** + **PB**.

THEOREM 35.

145. If the straight line joining two points subtends equal angles at two other points on the same side of it, the four points are concyclic.[1]

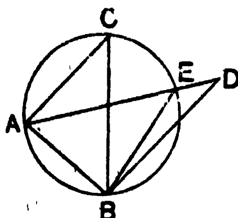


Fig. 347

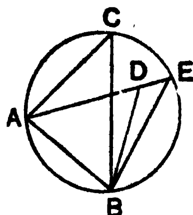


Fig. 348

Points **C** and **D** are situated on the same side of the line **AB**, such that ∠**ACB** = ∠**ADB**.

To prove that the four points **A,B,C,D**, are concyclic. Draw a circle to pass through the points **A, B, C** [see § 140]; then this circle must pass through **D**.

Proof. For, if the ⊙ **ABC** does not pass through **D**, it must cut **AD** (or **AD** produced) in some point **E** other than **D**.

Join **BE**.

Then ∠**AEB** = ∠**ACB**, being angles in the same segment. (*Theor.* 33)

But ∠**ADB** = ∠**ACB**, (*given*)

∴ ∠**AEB** = ∠**ADB**,

But this is impossible, for one of the two angles **AEB, ADB**, is an exterior angle, and the other an interior opposite angle of the △ **DEB**. (*Theor.* 8 § 65.)

1. For the meaning of the word *concyclic* see note 1, § 140.

Hence the ⊙ ABC must pass through D ;
that is, the points A, B, C, D are concyclic.

Corollary. **The locus of all points on one side of a given line BC, at which BC subtends angles equal to a given angle, is an arc of a circle with B and C for its end-points.**

[Find a point A such that ∠BAC = the given angle [see Fig. 239, § 104]. Draw a circle to pass through the points A, B, C ; then the arc BAC is the locus of all points on the same side of BC as A, at which BC subtends angles equal to the given angle (see Ex. 2, p. 303)].

THEOREM 36. [Euclid III. 22].

146. The opposite angles of any quadrilateral inscribed in a circle are supplementary.

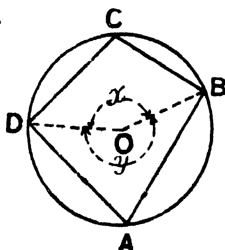


Fig. 349

ABCD is a quadrilateral inscribed in the ⊙ABC.

To prove that ∠BAD + ∠BCD = 2 rt. ∠s ;
and ∠ABC + ∠ADC = 2 rt. ∠s.

Join B and D to O, the centre of the ⊙.

Proof. $\angle BAD = \frac{1}{2} \angle x$, (*Theor.* 32)

$\angle BCD = \frac{1}{2} \angle y$

$\therefore \quad \angle BAD + \angle BCD = \frac{1}{2}(\angle x + \angle y)$

But $\angle x + \angle y = 4$ rt. $\angle$s.

$\therefore \quad \angle BAD + \angle BCD = 2$ rt. $\angle$s.

Similarly it may be shown that

$\angle ABC + \angle ADC = 2$ rt. $\angle$s.

Note 1. In Fig. 349 the quadrilateral taken is such that the centre O lies within it. In Fig. 350 the case is shown in which the centre lies outside the quadrilateral. The proof is the same in both cases.

Note 2. If a quadrilateral is such that its vertices lie on a circle, it is said to be **cyclic.**

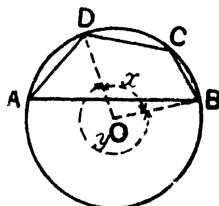


Fig. 350

Corollary. If one side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral.

ABCD is a cyclic quadrilateral, the side AB is produced to X. Then the ext. $\angle$ CBX = int. opp. $\angle$ ADC. [For, each of the $\angle$s CBX, and ADC is a supplement of the $\angle$ ABC (by Theor. 1 and Theor. 36).]

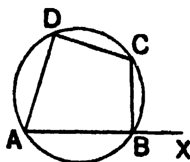


Fig. 351

THEOREM 37.

147. If a pair of opposite angles of a quadrilateral are supplementary, the qudrilateral is cyclic.[1]

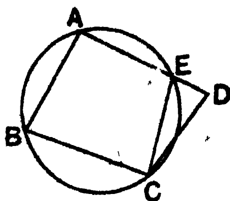


Fig. 352

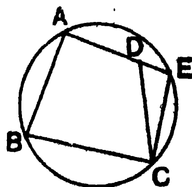


Fig. 353

ABCD is a quadrilateral in which ∠s ABC, ADC are supplementary.

To prove that the vertices A, B, C, D are concyclic.

Draw the circle passing through A, B, C. Then this circle must pass through D.

Proof. For if the circle ABC does not pass through D, it must cut AD (or AD produced) in a point E other than D. Then

$\angle AEC + \angle ABC = 2$ rt. ∠s; (*Theor.* 36)

but $\angle ADC + \angle ABC = 2$ rt. ∠s; (*given*)

$\therefore \angle AEC + \angle ABC = \angle ADC + \angle ABC.$

$\therefore \angle AEC = \angle ADC.$

But this is impossible, for one of these two angles is an exterior angle and the other an interior opposite angle of the △ CED. (*Theor.* 8, § 65.)

Hence the ⊙ ABC must pass through the point D, that is, the points A, B, C, D are concyclic.

1. See Note 2, § 146.

EXERCISE LX.

1. Find the locus of the mid-points of chords of a circle drawn through a fixed point [See Exs. 13, 14 (LIV)]

2 Find the locus of the intersection of the diagonals of a rhombus[1] one of whose sides is given in magnitude and position

3 **ABC** is a triangle inscribed in a circle Through **A** draw a chord which shall be bisected by **BC**. When does the problem fail?

4. **ABC** is a triangle and **PQ** a straight line Find a point (or points) in **PQ** at which **BC** subtends an angle equal to ∠**BAC** When does the problem fail?

5 **A** and **B** are two fixed points and **PQ** a given straight line Find a point (or points) in **PQ**, at which **AB** subtends an angle of given magnitude Discuss all possible cases

6 Given the base and the vertical angle of a triangle, show that its area is greatest when it is isosceles

7 Prove Theorem 36 by drawing the diagonals (without joining any of the vertices to the centre)

8 **ABCD** is a quadrilateral inscribed in a circle, and **AB**, **CD** are produced to meet at **P**, show that the △s **PAC**, **PBD** are equiangular, as also the △s **PBC**, **PAD**

9. **ABC** is an isosceles triangle Any line drawn parallel to the base **BC** cuts the equal sides **AB**, **AC** in **P**, **Q** Prove that the points **B**, **P**, **Q**, **C** are concyclic

10 **ABC** is a triangle and **BE**, **CF** are drawn perpendicular to **AC**, **AB**. Show that the points **B**, **E**, **C**, **F** are concyclic If **BE**, **CF** meet in **P**, show that the quadrilateral **AEPF** is cyclic, and ∠**BPC** is the supplement of ∠**BAC**.

---

1 Here the word 'rhombus' is to be understood in a more general sense so as to include a *square* as a particular case

11. Any parallelogram inscribed in a circle must be a rectangle.

12. Any trapezium inscribed in a circle must be isosceles. (See § 97).

* 13. Prove that if **ABCD** is an isosceles trapezium, the points **A, B, C, D** are concyclic.

14. Find the relation between the angles in the two segments into which a circle is divided by any chord.

15. **A** and **B** are two fixed points. A point **P** moves on one side and a point **Q** moves on the other side of the line **AB**, such that $\angle$ **APB** = 80° and $\angle$ **AQB** = 100°. Show that the loci of **P** and **Q** are arcs of the same circle.

16. **AOB, COD** are two intersecting lines such that **OA** = **OC**, and **OB** = **OD**. Prove that the points **A, B, C, D** are concyclic.

17. **ABC** is a triangle. The internal bisectors of $\angle$s **B, C** meet in **I**, and the external bisectors of the same angles meet in $\mathbf{I_1}$. Prove that $\mathbf{BICI_1}$ is a cyclic quadrilateral.

18. If two triangles have one side of the one equal to one side of the other and the angles opposite to those sides supplementary, prove that their circum-circles are equal.

19. **ADBC** is a circle (Fig. 354), **X** is a point on arc **ACB**; perpendiculars **AE, BF** are drawn to the sides **BX** and **AX** of the $\triangle$**AXB**, meeting in a point **P**. If **X** moves on the arc **ACB**, prove that the locus of **P** is an arc of a circle which is the image of the arc **ADB** with respect to the line **AB**.

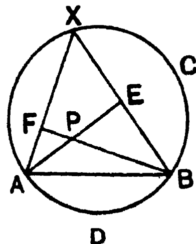


Fig. 354

20. **ABCD** is a parallelogram. A circle is drawn passing through the points **A** and **B**, so as

to cut **BC, AD** (produced if necessary) in **E** and **F.** Show that the four points **C, D, E, F** are concyclic.

21. **ABCD** is a quadrilateral inscribed in a circle. The bisector of ∠**ABC**, cuts the circle in **E**. Prove that **DE** is the external bisector of ∠**ADC.**

22. **ABCD** is a cyclic quadrilateral, the diagonals **AC, BD** intersecting in **O**. Show that the mid-points of **OA, OB, OC, OD** are concyclic.

23. **ABCD** is any quadrilateral. The bisectors of the ∠s **A, B ; B, C ; C, D ; D, A** meet successively in the points **P, Q, R, S.** Prove that the points **P, Q, R, S** are concyclic.

24. A chord **AB** divides the circumference into two arcs. **P** is any point on one of these arcs, and **Q** any point on the other arc. Show that the locus of the intersection of the bisectors of ∠s **PAQ, PBQ** is an arc of a circle passing through **A** and **B.**

25. **B, C, D** are three points on a circle whose centre is **O** such that ∠**BOD**=∠**BCD.** Prove that each of these angles must be 120°.

26. **ABCD** is a cyclic quadrilateral; **AB, DC** are produced to meet at **P**, and **AD, BC** to meet at **Q.** The circum-circles of the △s **PBC**, and **QAB** intersect again in **R.** Show that the points **P, Q, R** are collinear.

27. ABC is any triangle. AD, BE, CF, drawn perpendicular to BC, CA, AB respectively, meet in the point P [see § 135, IV]. Prove that the circum-circles of the △s BPC, CPA, APB, ABC are equal. [See Ex 18]

28. **ABC** is any triangle. On **AB, BC, CA** equilateral triangles **PBC, QCA, RAB** are drawn external to the triangle. Prove that the circum-circles of these △s pass through a common point. [Let the circum-circles of △s **PBC, QCA** meet again in **O**. Prove that **O** lies also on the circum-circle of the △ **RAB.**]

29. The straight lines which bisect any angle of a quadrilateral inscribed in a circle, and the opposite exterior angle meet on the circumference. [cf. Ex. 21]

30. **If L, M, N are the mid-points of the sides BC, CA, AB of a triangle ABC, and D is the foot of the perpendicular from A to BC, prove that D lies on the circle which passes through L, M, N. [∠MDN=∠BAC=∠MLN** ; [See Exs. 1, 5 (XXXIX) and Ex. 36 (Misc. Exercise I)].

31. **If L, M, N are the mid-points of the sides of a triangle and D, E, F are the feet of the perpendiculars from the vertices to the opposite sides, prove that the six points L, M, N, D, E, F lie on a circle.**

EQUAL ARCS, ANGLES AND CHORDS IN THE SAME CIRCLE OR EQUAL CIRCLES.

THEOREM 38. [Euclid III, 26, 27].

148. In equal circles (or in the same circle)

(i) if two arcs subtend equal angles at the centres, they are equal ;

conversely, (ii) if two arcs are equal they subtend equal angles at the centres.

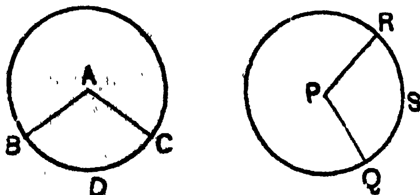


Fig. 355

(i) **BDC** and **QSR** are equal circles whose centres are **A** and **P** respectively.

Arcs **BDC**, **QSR** subtend equal angles **BAC**, **QPR** at the respective centres **A**, **P**.

To prove that arcs **BDC**, **QSR** are equal.

Proof. Apply ⊙ **QSR** to ⊙ **BDC**, so that centre **P** may fall on centre **A**, and **PQ** along **AB**.

Then since ∠ **QPR** = ∠ **BAC**, (*Given*)

∴ **PR** will fall along **AC**.

Since the ⊙ˢ have equal radii, **Q** will fall on **B**, and **R** on **C**, and the circumferences of the two ⊙ˢ will coincide entirely.

∴ arc **QSR** will coincide with arc **BDC** ;

∴ arcs **QSR** and **BDC** are equal.

(ii) **The converse Theorem :**

BDC, **QSR** are equal ⊙ˢ whose centres are **A**, **P** respectively. (Fig. 355).

Arcs **BDC**, **QSR** are equal.

To prove that ∠ˢ **BAC**, **QPR** subtended by these arcs at the respective centres are equal.

Proof. Apply ⊙ **QSR** to ⊙ **BDC**, so that centre **P** may fall on centre **A**, and **PQ** along **AB**.

Then since the ⊙ˢ have equal radii, **Q** falls on **B**, and the two circumferences coincide entirely.

Since arc **QSR** = arc **BDC**, **R** falls on **C**.

Thus **PQ** coincides with **AB**, and **PR** with **AC** ;

hence ∠ **QPR** coincides with, and is therefore equal to ∠ **BAC**.

Note 1. The *sectors* **PQR** and **BAC** are equal.

Note 2. The proposition (Theor. 38) is proved above for equal circles. If the ⊙s **BDC**, and **QSR** were superposed, they would become one circle, and the arcs **BDC**, **QSR** would become arcs of the same circle. This consideration shows how the proposition holds for arcs and angles in the same circle.

Corollary 1. *In equal circles (or in the same circle),*

(*i*) *if two arcs subtend equal angles at points on the remaining parts of the circumferences, they are equal* [Fig. 356.]

(ii) conversely, *if two arcs are equal they subtend equal angles at points on the remaining parts of the circumferences.*

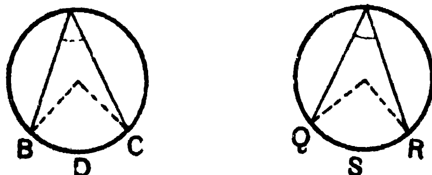


Fig. 356

Corollary 2. *If two arcs of a circle subtend unequal angles at the centre, they are unequal, the arc subtending the greater angle being the greater. Conversely, the greater of two arcs of the same circle subtends the greater angle at the centre.*

Corollary 3. *If an arc of a circle is p times another, the angle subtended at the centre by the first arc is p times the angle subtended by the second.*

THEOREM 39. [Euclid III. 28, 29].

149. In equal circles (or in the same circle)

(i) if two chords are equal, they cut off equal arcs ; conversely, (ii) if two arcs are equal, the chords which cut off these arcs are equal.

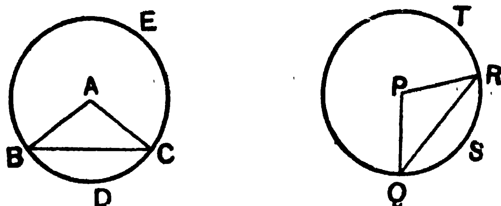


Fig. 357

(i) BDC, QSR are two equal ⊙ˢ whose centres are A, P respectively.

Chord BC = chord QR.

To prove that minor arc BDC = minor arc QSR, and major arc BEC = major arc QTR.

Join AB, AC, PQ, PR.

Proof. In the △ˢ BAC, QPR,

BC = QR,	(*given*)
AB = PQ,	(radii of equal ⊙s)
AC = PR,	(„ „ „)

∴ the △ˢ are congruent,

∴ ∠BAC = ∠QPR ;

∴ arc BDC = arc QSR. [*Theor.* 38 (i)]

Again the whole circumference of ⊙ BDC = whole circumference of ⊙ QSR ;

∴ the remaining arc BEC = the remaining arc QTR.

(ii) The converse theorem,

BDC, QSR are two equal ⊙s whose centres are A, P respectively. (Fig. 357).

Arc BDC = arc QSR.

To prove that chord BC = chord QR.

Join AB, AC, PQ, PR.

Proof. Since arc BDC = arc QSR, (*given*)

∴ ∠BAC = ∠QPR. [*Theor.* 38 (ii)]

Now in the △s BAC, QPR,

AB = PQ (radii of equal ⊙s),
AC = PR („ „ „),
∠BAC = ∠QPR, (*proved*)

∴ the △s are congruent,

∴ chord BC = chord QR.

An application. To bisect a given arc of a circle.

To bisect the arc AOB (Fig. 358).

Join AB and draw DO the perpendicular bisector of AB, meeting the arc in O. Then the arc is bisected at O. For, O being a point on the perpendicular bisector of AB,

AO = BO, (Theor. 27)

∴ arc AO = arc BO. [*Theor.* 39 (i)]

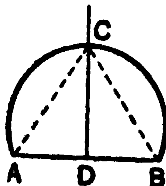


Fig. 358

EXERCISE LXI.

1. **A, C, B** are three points on a circle. Show that the bisector of ∠**ACB** bisects the arc **AB** on which the angle stands.

2. **A, C, B** are three points on a circle such that **AC**=**CB**. Show that **C** is the mid-point of the arc **ACB**, and **OC** bisects the chord **AB** at right angles (**O** being the centre of the ⊙).

3. The arcs intercepted by two parallel chords of a circle are equal.

4. **ABCD** is a quadrilateral inscribed in a circle such that **AB**=**CD**. Show that **AC**=**BD**, and **AD**, **BC** are parallel

5. Two equal circles intersect in **A** and **B**. Any line **PAQ** is drawn through **A**, cutting the circumferences in **P** and **Q**. Prove that **PB**=**QB**

6. **AB** *is a fixed chord of a circle and* **P** *any point on one of the arcs cut off by* **AB**. *Prove that the bisector of* ∠**APB** *passes through a fixed point on the circle, for all positions of* **P**.

7. **P** is any point on the arc **AB** of a circle; **AP** is produced to any point **Q**, prove that the bisector of the angle **BPQ** bisects the arc **AB**.

8. **B, C, D** are three points on a circle whose centre is **O**, such that ∠**BOD**=∠**BCD**. Show that the arc **BCD** is half the arc conjugate to it [See Ex 25, (Exercise **LX**)]

9. **A** *and* **B** *are two fixed points and* **P** *a point on a specified side of the line* **AB**, *such that the* ∠**APB**=*a given angle* p. *A line* **PQ** *is drawn through* **P** *within the* ∠**APB**, *such that the* ∠**APQ**=*another given angle* q *(less than* p*). Show that* **PQ** *passes through a fixed point, for all positions of* **P**, [Cf. Ex. 6.]

10. If the arc **AB** of a circle is double the arc **CD** of the same circle, prove that the chord **AB** is less than twice the chord **CD**.

11. **AB, CD** are two chords of a circle such that the chord **CD** is half the chord **AB**; prove that the arc **CD** is less than half the arc **AB**.

12. Two equal circles intersect in **A** and **B**; any line **PAQ** is drawn through **A** cutting the circumferences in **P** and **Q**; find the locus of the mid-point of **PQ**. [The circle with **AB** as diameter; see Ex, 5].

13. Through any point **A** within a circle two chords **PAQ, RAS** are drawn at right angles to each other. Show that arc **PR**+arc **SQ**=arc **PS**+arc **QR**.

14 Repeat the examples on pp. 57, 58, and Exs. 9–12 of Exercise VIII.

150. Intersection of a circle and a straight line. Tangent to a circle.

Let **AB** be a straight line of unlimited length, and **O** any point outside it. Draw **ON** perpendicular to **AB**. In Theorem 12 (§ 69) it has been proved that of all straight lines from **O** to **AB**, **ON** is the shortest; if **P, Q** are two points on **AB**, on opposite sides of **N** but equidistant from it, **OP** = **OQ**, and no other straight line from **O** to **AB** can be equal to **OP** or **OQ**. [See Note 1, § 69].

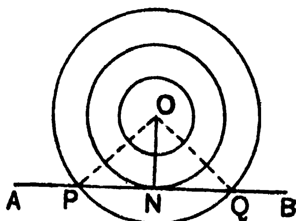


Fig. 359

It follows that a circle drawn with centre **O** will meet **AB** in *two points*, in *one point*, or *in no point*, according as the radius of the circle is *greater* than,

equal to, or *less* than **ON** [fig. 359]. In other words a straight line will meet a circle in *two points*, in *one point*, or in *no point* according as the perpendicular distance of the centre of the circle from the line is *less* than, *equal* to, or *greater* than the radius of the circle.

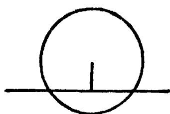


Fig. 360

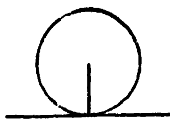


Fig. 361

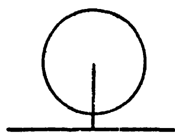


Fig. 362

When a straight line cuts a circle in two points as in Fig. 360, it is called a **secant**, the portion of the line intercepted within the circle being a chord.

When a straight line meets a circle in a point, *without cutting it*, as in Fig. 361, it is said to **touch** the circle at the point, and is called a **tangent**. The point at which a tangent touches the circle, is called the **point of contact.** Note that *every point in a tangent except the point of contact is outside the circle.*

Another point of view. Consider a secant **AB** cutting the circle (Fig. 363) in two points **P, Q.** Suppose the secant to move parallel to itself as shown in the figure. The points of intersection **P, Q,** come nearer and nearer as the secant moves further

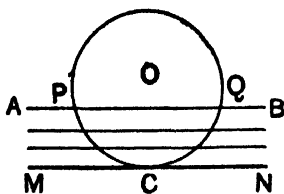


Fig. 363

and further from the centre ; and when the secant comes to the position **MN,** the points **P** and **Q** meet, and

in this limiting position the secant becomes a tangent to the circle.

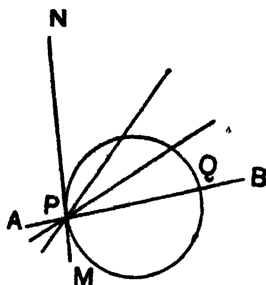


Fig 364

If we had supposed the secant AB to turn round the point P (as shown in Fig. 364) instead of moving parallel to itself, then also we would obtain the tangent MN as the limiting position of the secant, having P for the point of contact.

THEOREM 40. [Euclid III. 18]

151. **The tangent at any point of a circle and the radius through the point are perpendicular to one another.**

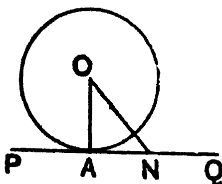


Fig 365

PQ is the tangent to a circle (centre O) at the point A.

To prove that the tangent PQ and the radius OA are perpendicular to one another.

Proof. If OA is not ⊥ to PQ, suppose ON is the perpendicular from O to PQ.

Since ON is ⊥ to PQ,

∴ ON < OA (a radius of the ⊙), (*Theor.* 12)

∴ the point N is inside the circle [see § 136, (i)].

But this is impossible, for PQ being a tangent, every

point in it except A (the point of contact) is *outside* the circle. ∴ OA is ⊥ to PQ.*

Cor. 1. The perpendicular to a tangent at the point of contact passes through the centre. [This result follows from Theorem 40, and the fact that from a given point only one perpendicular can be drawn to a given straight line.]

Cor. 2. If through a point A on a circle a straight line AQ, is drawn perpendicular to the radius through A, then AQ is the tangent to the circle at the point A. Note that *only one tangent can be drawn to a circle at a given point* A *on it* (for only one perpendicular can be drawn at A to the radius through A).

Note. The perpendicular drawn to a tangent at the point of contact is called the **normal** at the point. It is evident that *all normals to a circle are concurrent*, as they all pass through the centre.

---

* **Alternative Proof.** Let **APQ** be any secant through **A**, and **AOB** the diameter through **A**. Then ∠**APB** = a right angle [Theor 34]. ∴ ∠**QPB** is a right angle.

Let the secant **APQ** turn round **A** as shown in the figure. The point **P** comes nearer and nearer to **A**, and when the secant **APQ** comes to the position **AT**, the point **P** comes to the point **A**, and **AT** is the tangent at **A**; ∠**QPB** now becomes ∠**TAB** which is therefore a right angle (for in every position of the secant ∠**QPB** is a right angle), that is to say, *the tangent at any point* **A** *and the radius through* **A** *are at right angles.*

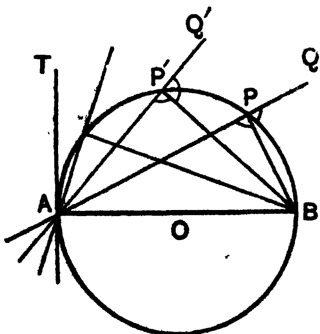


Fig. 365 (a)

THEOREM 41.

152. (i) Two tangents can be drawn to a circle from an external point.

(ii) The two tangents to a circle from an external point are equal, and they subtend equal angles at the centre.

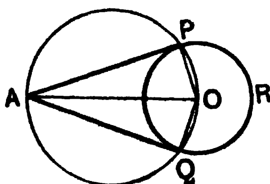


Fig. 366

PQR is a ⊙ whose centre is **O**, and **A** is a point outside the circle.

(i) *To prove that* two tangents can be drawn to the circle from **A**.

Construction. Join **OA**, and draw the circle **OPAQ** on **OA** as diameter, cutting the given circle in two points **P** and **Q**. Join **AP, AQ**.

Then **AP, AQ** are tangents to the circle **PQR**.

Proof Join **OP, OQ**.

Since **OA** is a diameter of the ⊙ **OPAQ**,

∴ each of the ∠s **APO** and ∠ **AQO** is a right angle. (*Theor.* 34)

∴ **AP, AQ** are tangents to the circle **PQR**. [*Theor.* 40, *Cor.* 2]

Thus there are two tangents from **A** to the circle.

(ii) *To prove that* the tangents **AP**, **AQ** are equal, and the $\angle$s **AOP**, **AOQ** are also equal.

Proof. In the $\triangle$s **AOP**, **AOQ**,

$\angle$s **APO**, **AQO** are rt. $\angle$s,
hypotenuse **AO** is common,
OP = **OQ** (radii of the same circle),

$\therefore$ the $\triangle$s are congruent, [*Theor.* 18]

$\therefore$ **AP** = **AQ**

and $\angle$**AOP** = $\angle$**AOQ**.

Corollary :—**AO** *bisects the angle* **PAQ** *between the tangents from* **A**.

Ex. *Can a circle be drawn to touch three concurrent straight lines ?*

Note 1. To calculate the length of the tangents AP, AQ (Fig. 366).

Since $\angle$**APO** is a right angle, we have by Pythagoras's Theorem,

$$AO^2 = AP^2 + OP^2$$
$$\therefore\ AP^2 = AO^2 - OP^2$$

If the radius of the circle $= a$ units of length, the distance of **A** from the centre **O**, (*i.e.* **OA**) $= d$ units of length, and the tangent **AP** or **AQ** $= l$ units of length we have,

$$l^2 = d^2 - a^2,$$
$$\therefore\ l = \sqrt{d^2 - a^2}$$

Note 2. If a circle and a polygon are such that each side of the polygon is a tangent to the circle, then the circle is said to be **inscribed** in the polygon. and the polygon is said to be **circumscribed** about the circle (Fig. 367)

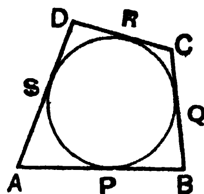


Fig. 367

Observe that **AS** = **AP**, **BP** = **BQ**, **CQ** = **CR**, **DR** = **DS**.

153. Construction of tangents to a circle. We have seen that only one tangent can be drawn at a point *on the circle*, but two tangents from a point *outside the circle*. It may be noted that *no* tangent can be drawn from a point *within the circle*.

Cor. 2 of Theorem 40, indicates a method of drawing the tangent at a given point *on the circumference*; the method of constructing the *two* tangents from an *external* point is indicated in Theor. 41

Ex 1. Draw a circle of radius 1 inch, and take a number of points on the circumference construct the tangents at those points.

Ex. 2. Draw two circles intersecting in two points **A**, **B** Draw the tangents to the circles at **A** and **B**, and measure the angle between the tangents drawn at **A**, and also that between the tangents drawn at **B**.

Ex. 3. Draw a circle of radius 5 cm., and take a point **A** 9 cm from the centre Construct the tangents from **A** to the circle.

Ex 4. Draw two circles intersecting in two points **A** and **B**. Take any point **P** on **AB** (or **BA**) *produced*, and construct the tangents from **P** to the two circles Measure the four tangents, do you find them equal in length ?

EXERCISE LXII.

1. Prove that the tangents drawn to a circle at the extremities of a chord make equal angles with the chord

2, Find the locus of centres of circles which touch a given straight line at a given point in it.

3 Find the locus of the centres of circles of a given radius which touch a given straight line.

4. Find the locus of centres of circles which touch the arms of a giveen angle **BAC**. [See Cor , Theor 41].

5. **BAC** is an angle of 40°. Construct a circle of radius 1 inch within the angle **BAC** having **AB**, **AC** for tangents.

6. Find the locus of the centres of circles which touch two intersecting straight lines of unlimited lengths.

7. Calculate the length of the tangents drawn to a circle of diameter 10 cm. from a point 8 cm. from the centre.

8. The length of a tangent drawn to a circle of radius 2·5 inches from an external point **P** is 5 inches. What is the distance of **P** from the centre of the circle ?

9. The length of a tangent drawn to a circle from a point 10 cm. from its centre is 8 cm. Calculate the radius of the circle.

10. **P** is any point on a circle, and from the tangent at **P** a length **PT** is cut off equal to a given length. Find the locus of **T**.

11. Two concentric circles have radii 4 in. and 6 in.; calculate the length of a chord of the outer circle which is a tangent to the inner.

12. Prove that all chords of a given circle, which touch an inner concentric circle are equal.

13. Prove that all chords of a circle which are of a given length touch a concentric circle.

14. Two tangents can be drawn to a circle parallel to any given straight line, and the points of contact are the extremities of a diameter.

15. Draw two tangents to a given circle, which shall include a given angle.

16 Cicrumscribe a parallelogram about a given circle. How many such parallelograms are possible ?

17. Prove that any parallelogram circumscribed about a circle is a rhombus.

18. **If a quadrilateral is circumscribed about a circle, show that the sum of one pair of opposite sides is equal to the sum of the other pair of opposite sides.**

19. If a quadrilateral is circumscribed about a circle the angles subtended at the centre by a pair of opposite sides are supplementary.

20. If a polygon circumscribes a circle, prove that the area of the polygon is half the product of the radius of the circle and the perimeter of the polygon. (Divide the polygon into a number of triangles by joining the centre of the circle to lhe vertices of the polygon).

154. Intersection of circles. Contact of circles. We have seen that two circles connot meet in more than two points (see § 140, Cor. 2).

Two circles may meet in two points, as in Fig. 368; they may meet in one point, as in Figs. 369, 370; or they may not meet, as in Figs. 371, 372. In Figs, 369, 370

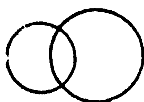


Fig. 368

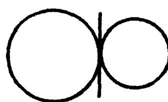


Fig. 369

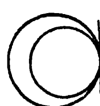


Fig. 370

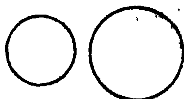


Fig. 371

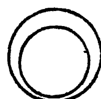


Fig. 372

the circles meet at a point but do not *cut* each other. *They have the same tangent line at the point of meeting.*

When two circles meet at a point, and have the same tangent line at the point, they are said to

be in contact, or touch each other, at the point, and the point is called **the point of contact.** Two circles touch each other **externally,** when they are on *opposite* sides of the common tangent, as in Fig. 369, and **internally,** when they are on the *same* side of the common tangent, as in Fig. 370.

If two circles *cut* each other at a point, without *touching,* **the angle at which they cut** is defined to be the angle between the tangents to the two circles drawn at that point. This angle is also equal to the angle between the radii[1] of the two ⊙s drawn through the point of intersection.

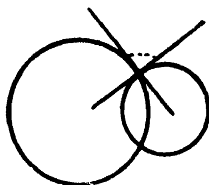


Fig. 373

Two circles are said to be **orthogonal,** or to cut each other **orthogonally,** if they cut at right angles, that is to say, if the tangents to the circles at the point of intersection are at right angles. [See Ex. 2 (LXIII)].

In the same way, if a straight line cuts a circle at a point A, **the angle at which it cuts the circle** is defined to be the angle which it makes with the tangent to the circle at A. [See Ex. 1 (LXIII)].

Ex. 1. *If two circles cut orthogonally at* **P**, *the angle* **OPO′** *is a right angle, where* **O, O′** *are the centres of the circles.*

Ex. 2. *If two circles cut orthogonally at* **P**, *the radius through* **P** *of one circle is a tangent to the other at* **P**.

Ex. 3. **If two circles cut orthogonally then the sum of the squares on their radii is equal to the square on the distance between their centres. State and prove the converse of this result also.**

---

1. For, the radii are perpendicular to the respective tangents see Ex. 13, p. 200.

THEOREM 42. [Euclid III, 12].

155. **If two circles touch, the point of contact lies in the straight line through the centres.**

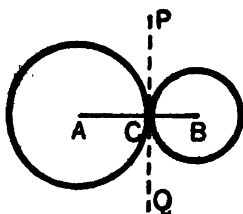


Fig. 374

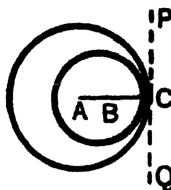


Fig. 375

Two ⊙s with centres A and B, touch externally (Fig. 374), or internally (Fig. 375) at C.

To prove that the point of contact C lies in the straight line passing through A and B.

Join AC, BC.

Proof. Since the ⊙s touch at C, they have a common tangent PQ at C. (see § 154).

Since PQ is a tangent to both the ⊙s at the point C,
∴ PQ is ⊥ to each of the radii CA, CB. (*Theor.* 40)
∴ ∠s PCA, PCB are each a right angle.

Now if the ⊙s touch exernally (Fig. 374),
∠PCA + ∠PCB = 2 rt. ∠s;
∴ ACB is one straight line. (*Theor.* 2).

And if the ⊙s touch internally (Fig. 375),
∠PCA is equal to ∠PCB, being each a right angle,
∴ CBA is one straight line.

Cor. 1. If two circles touch externally, the distance between their centres is equal to the sum of their radii; if they touch internally the distance between their centres is equal to the difference of their radii.

Cor. 2. If two circles meet at a point C, and C lies on the straight line through the centres, then the circles touch at C. [The ⊙s have a common tangent at C]. From this the following result follows.

Cor. 3. *If the distance between the centres of two ⊙s is equal to the sum of their radii, they touch externally; if the distance between the centres is equal to the difference of the radii, the circles touch internally.*

EXERCISE LXIII.

1. If a straight line cuts a circle in two points, it cuts the circle at the same angle at each of these points. [See Ex 1, (LXII)].

2 *If two circles intersect in two points, they cut each other at the same angle at each of these points.*

3 Draw a circle which cuts a given circle at a given point in it, at an angle of 60°. How many such circles can you draw?

4 Two circles of radii 10 cm. and 12 cm. touch externally, what is the distance between their centres?

5. In Ex. 4, if the circles touch internally, what is the distance between the centres?

6. The radii of two circles are r, R and the distance between

their centres is d. Find whether the circles *touch*, or *cut*, or *neither touch nor cut* in the following cases.

(i) $r=3$ cm., $R=5$ cm., $d=10$ cm.
(ii) $r=3$ cm, $R=5$ cm, $d=8$ cm.
(iii) $r=3$ cm, $R=5$ cm, $d=6$ cm.
(iv) $r=3$ cm, $R=5$ cm., $d=4$ cm
(v) $r=3$ cm, $R=5$ cm, $d=2$ cm.
(vi) $r=3$ cm, $R=5$ cm, $d=1\cdot5$ cm.
(vii) $d=r+R$. (viii) $d>r+R$.
(ix) $d=R-r$. (x) $d<R-r$.

7. Draw four circles to touch a given circle at a given point. What is the locus of the centres of circles which touch a given circle at a given point ?

8. Draw four circles of a given radius to touch a given circle externally What is the locus of the centres of circles of a given radius which touch a given circle externally ?

9. Draw four circles of a given radius to touch a given circle internally. What is the locus of the centres of circles of a given radius which touch a given circle internally ?

10. Three circles of radii 3, 4, 5 cm. are such that each touches the other two externally. What are the distances between the centres ? Draw the circles

11. **ABC** is an equilateral triangle of side 6 cm Draw three circles having **A**, **B**, **C** for centres, so that each circle may touch the other two externally.

12 Two circles whose centres are **A** and **B** touch at **C**. Through **C** any straight line is drawn cutting the circles in **P** and **Q** respectively; prove that the radii **AP** and **BQ** are parallel.

13. In Ex. 12, show that the tangents to the two circles at **P** and **Q** are parallel.

14. Two given circles whose centres are **A** and **B** lie one within the other; **P** is the centre of any circle which touches the larger of the given circles internally, and the smaller externally. Show that, **AP**+**BP** is constant. What is the locus of **P** ? [See Ex. 1, § 131].

15. A variable circle of centre **P**, touches two fixed circles of centres **A** and **B** externally Show that **PA** − **PB** is constant What is the locus of **P** ? [See Ex. 15, § 131].

THEOREM 43. [Euclid III, 32]

156. **If a straight line touch a circle and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments of the circle.**

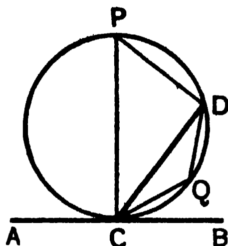


Fig. 376

AB touches the ⊙ PCD at C, and through C a chord CD is drawn.

To prove that

(i) ∠DCB = angle in the alternate segment DPC;

(ii) ∠ACD = angle in the alternate segment CQD.

Through C draw the diameter CP.

Join DP.

Proof. (i) Since CP is a diameter,

∴ ∠CDP is a rt. ∠; (*Theor.* 34)

∴ ∠PCD + ∠DPC = 1 rt. ∠. [*Theor.* 16, Note 3 (iii)]

Again since ACB is the tangent at C, and CP the diameter through C,

∴ ∠PCB is a rt. ∠, (*Theor.* 40)

that is, $\angle PCD + \angle DCB = 1$ rt. $\angle$.

$\therefore \angle PCD + \angle DCB = \angle PCD + \angle DPC$.

$\therefore \angle DCB = \angle DPC$.

(ii) Take any point Q on the arc CQD,

Join CQ, DQ.

Since CPDQ is a cyclic quadrilateral,

$\therefore \angle CPD + \angle CQD = 2$ rt. $\angle$s. (*Theor.* 36)

Again $\angle BCD + \angle ACD = 2$ rt. $\angle$s, (*Theor.* 1)

$\therefore \angle BCD + \angle ACD = \angle CPD + \angle CQD$;

but $\angle BCD = \angle CPD$, (proved)

$\therefore \angle ACD = \angle CQD$.

Note. The converse of the above theorem is true :

If at an extremity **C** *of a chord* **CD** (Fig. 376) *of a circle a line* **CB** *is drawn making* $\angle$ **DCB** = *angle in the alternate segment* **CPD**, *then* **CB** *is the tangent to the circle at* **C**.

The proof is left to the student.

EXERCISE LXIV.

1. In fig. 376, if $\angle BCD = 50°$, find $\angle$s **CPD, ACD, CQD**.

2. Find all the angles in Fig. 377.

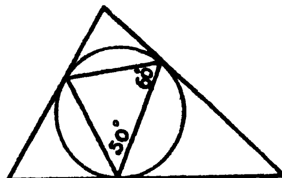


Fig. 377

3. Prove by Theorem 43 that the tangents to a circle from an external point are equal. [Join the points of contact].

4. **AB** is a chord of a circle. A tangent is drawn parallel to **AB**, touching the circle at **C**. Show that **C** is the mid-point of the arc **ACB**.

5. Two circles touch internally at **A**, and two straight lines **APQ ARS** are drawn through **A** cutting the circles in **P, Q** and **R, S** respectively. Prove that **PR** and **QS** are parallel.

6. Two circles touch externally at **A**, and two straight lines **PAQ, RAS** are drawn cutting the circles in **P, Q** and **R, S** respectively. Prove that **PR** and **QS** are parallel.

7. **ABC** is a triangle. A point **P** is taken in **BC** such that ∠**APC** = ∠**BAC**. Show that **AC** touches the circumcircle of the **APB**.

8. **BC** is a chord of a circle, and **BT** is the tangent at **B**. Show that the bisectors (internal and external) of ∠**CBT** bisect the arcs cut off by **BC**.

9. **ABC** is a triangle inscribed in a circle. Any line is drawn parallel to the tangent at **A**, cutting **AB**, **AC**, in **P** and **Q**. Show that **B**, **C**, **Q**, **P** are concyclic.

10. Two circles touch internally at **A**. A straight line **PQRS** cuts the circles in **P**, **Q**, **R**, **S** in order. Show that **PQ**, **RS** subtend equal angles at **A**.

11. Two circles touch externally at **A**. A straight line **PQRS** cuts the circles in **P**, **Q**, **R**, **S** in order. Show that **PS** and **QR** subtend supplementary angles at **A**.

12. Two circles touch each other; show that any straight line drawn through the point of contact cuts off similar segments from the circles.

13. Two circles intersect in **A**, **B**; and **PAQ** is any straight line through **A** cutting the ⊙s in **P** and **Q**. If the tangents to the two ⊙s drawn at **P** and **Q** intersect in **R**, prove that the four points **B**, **P**, **Q**, **R** are concyclic.

14. Two circles touch internally at **A**. A chord **CD** of the larger circle touches the smaller at **E**, prove that **AE** bisects ∠**CAD**. [See Ex. 10]

15. **ABC** is a triangle. The bisector of ∠**A** cuts **BC** in **D**, and the perpendicular bisector of **AD** cuts **BC** produced in **E**. Prove that **AE** is a tangent to the circumcircle of the △ **ABC**. Interpret the case when the △ **ABC** is isosceles.

CHAPTER XIII.

PROBLEMS.

157. Construction of the common tangents to two given circles.

(*a*) Direct Common Tangents.

Let us suppose the circles to be unequal, and let **A** be the centre of the larger ⊙, and **B** that of the smaller.

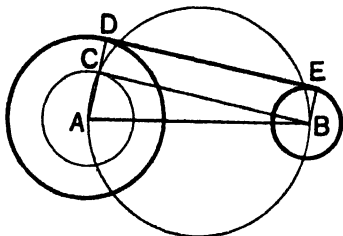


Fig. 378

Construction. With centre **A** and radius equal to the difference of the radii of the given ⊙s, describe a circle

From **B** draw a tangent **BC** to this circle[1]. Join **AC** and produce it to meet the larger given circle at **D**.

From **B** draw radius **BE** ∥ to **AD**. (in the direction **AD**, as shown in Fig. 378).

Join **DE**, then **DE** is a common tangent to the two given ⊙s.

Proof. Since **CD** and **BE** are equal[2] and parallel,

∴ **CDEB** is a parallelogram.

Now ∠**ACB** is a right angle [*Theor.* 40).

∴ ∠**BCD** is a right angle.

∴ **CDEB** is a rectangle.

∴ ∠s **ADE, BED** are each a right angle.

∴ **DE** is a tangent to each circle[3]. (Theor. 40, Cor. 2).

Note that **DE**=**BC**

---

1. See § 153. A circle is described with **AB** as diameter (as shown in Fig. 378). This circle cuts the circle (centre **A**) to which a tangent from **B** is to be drawn, in two points of which one is **C**; then **BC** is a tangent from **B** to the circle in question.

2. Since **AC**=difference of the radii of the given circles, **AC**=**AD**−**BE**, also **AC**=**AD**−**CD**. ∴ **BE**=**CD**.

3. The clue to the construction is obtained by Analysis. See footnote, p. 239.

Note. Since another tangent can be drawn from **B** to the inner circle with centre **A**, another common tangent of the same type as **DE** can be drawn to the two given ⊙s. Construct this second common tangent.

It should be noted that both the circles lie on the same side of the common tangent **DE**. When a common tangent to two ⊙s is such that both the ⊙s lie on the *same* side of it, it is called a **direct common tangent**. Thus **DE** is a direct common tangent. There are two such tangents to every pair of circles, provided the circles are such that one does not lie wholly within the other.

In the above construction of a direct common tangent, the circles were supposed to be unequal. If the ⊙s were equal a direct common tangent might be constructed very easily as follows :—

Join **AB**. At **A**, **B**, draw ⊥s to **AB**, meeting the ⊙s in **D** and **E**. Join **DE**. Then **DE** is a direct common tangent. The proof is left to the student.

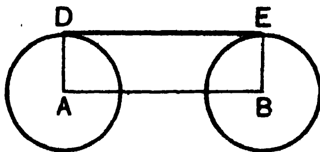


Fig. 379

(*b*) **Transverse common tangents.** If two circles are such that they lie wholly outside each other, a common tangent can be drawn to them, *so that the circles will lie on opposite sides of it.* Such a common tangent is called a **transverse common tangent.** The method of constructing such a common tangent is given below. **A** and **B** are

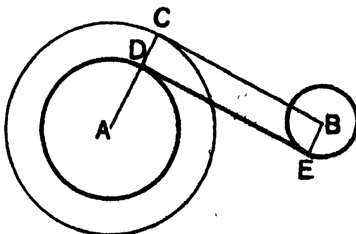


Fig. 380

the centres of the two given circles. With centre **A** and radius = the *sum* of the radii of the given ⊙s, describe a circle, and from **B** draw a tangent **BC** to this circle.

Join **AC** and let **AC** cut the *given* ⊙ (centre **A**) at **D**. Through **B** draw radius **BE** || to **CA**, (in the direction **CA** as shown in Fig. 380).

Join **DE**, then **DE** is a common tangent to the given ⊙s. The proof is left to the student.*

The given circles lie on opposite sides of **DE** (Fig. 380), and **DE** is a *transverse* common tangent. Since another tangent can be drawn from **B** to the outer circle with centre **A**, another transverse common tangent can be drawn to the two circles. Construct this tangent.

Note If two circles lie wholly outside each other, *four* common tangents can be drawn to them—two, direct; and two, transverse.

If two circles cut each other at two points, only two common tangents can be drawn to them and both are direct. Note also the three cases; (i)—when the ⊙s touch each other externally (ii) when they touch internally (iii) when one of the ⊙s lies completely within the other.

Ex. 1. Draw two circles of radii 3 cm. and 4 cm. with their centres 6 cm. apart. Construct the two direct common tangents. Calculate the length of each of these tangents (i.e. the portion, intercepted between the points of contact), and verify by measurement.

Ex. 2. Have the circles drawn in Ex. 1 any transverse common tangent? Show that the construction for transverse common tangents fails in this case.

Ex. 3. Draw two circles of radii 2 cm. and 3 cm. with their centres 6 cm. apart. Draw the two transverse common tangents. Calculate the lengths of these tangents and verify by measurement.

Ex. 4. In Ex. 6.. (Exercise LXIII) find the number of common tangents (if any) to the two circles in each of the cases, (i), (ii)......(vi) and construct the common tangents.

Ex. 5. **If two direct common tangents be drawn to two circles, prove that the lengths of these tangents (i.e. portions intercepted between the points of contact) are equal.**

Ex. 6. **Prove that the two transverse common tangents to two circles are equal in length**

Ex. 7. **Prove that the direct common tangents to two circles intersect on the straight line joining the centres; so also do the two transverse common tangents.**

---

* Prove that **BCDE** is a rectangle, so that **DE** is ⊥ to **BE** and **AD**, and hence a tangent to both the circles.

158. To inscribe a circle in a given triangle* (Euclid IV. 4)

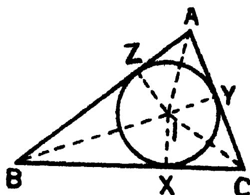


Fig. 381

Let ABC be the given triangle (Fig. 381). Draw the bisectors of ∠ˢ B, C and let them meet in I. Then I is equidistant from BC, CA, AB ; and AI is the bisector of ∠A [§ 135 II (*a*)]. Draw IX, IY, IZ ⊥ to BC, CA, AB respectively.

Since IX=IY=IZ a circle described with centre I and radius IX will pass through X, Y, Z. Draw this circle. It is the inscribed ⊙ of the △ ABC.

Proof. Since IX, IY, IZ are radii of the circle, and BC, CA, AB are ⊥ to IX, IY, IZ respectively,

∴ BC, CA, AB touch the circle at X, Y, Z.

Note 1. AY=AZ, BZ=BX, CX=CY. [*Theor.* 41 (ii)].
If a, b, c are the lengths of BC, CA, AB, and $2s=a+b+c$ (i.e., the perimeter of the △), it may be shown that AY=AZ=$s-a$, BZ=BX=$s-b$, CX=CY=$s-c$.

Note 2. △ABC=△IBC+△ICA+△IAB=½ IX. BC+½ IY. CA+½IZ. AB. [see § 113]. If a, b, c, be the lengths of the sides BC CA, AB, $2s$ the perimeter, S the area of the △ABC, and r the radius of the inscribed circle, we have

$$S=\tfrac{1}{2}r\ (a+b+c)=\tfrac{1}{2}\ r\ (2s)=rs.$$
$$r=S/s$$

Note 3. *The circles whose centres are* A, B, C, *and radii* AY BZ, CX *respectively touch one another externally* (*see Cor.* 3, *Theor.* 42).

---

* The problem of constructing the circumscribed circle of a triangle has been already considered. [see Cor. 4, § 140, also § 135, I].

159. To draw an escribed circle of a triangle, that is, a circle which touches one side of the triangle, and the other two sides produced.

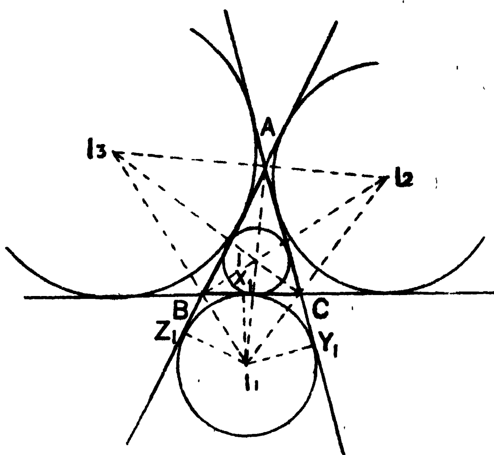


Fig. 382

Let ABC be a triangle (Fig. 382). Draw the external bisectors of $\angle^s$ B, C, and let them meet in I_1. Draw I_1X_1, I_1Y_1, I_1Z_1 ⊥ to BC, CA, AB respectively;

then $I_1X_1 = I_1Y_1 = I_1Z_1$. [see § 135, II (b)].

∴ a circle described with centre I_1 and radius I_1X_1 will pass through X_1, Y_1, Z_1:

and since I_1X_1, I_1Y_1, I_1Z_1 are respectively ⊥ to BC, CA, AB, this ⊙ touches BC at X_1, and AC,AB *produced* at Y_1, Z_1. It is thus an escribed circle of the triangle ABC. There are two other escribed ⊙^{s} with centres I_2, I_3, as shown in Fig. 382.

Note 1. If I be the centre of the inscribed circle (Fig. 382), it will be easily seen that I, I_1 lie on the internal bisector of ∠A;

I, I_2 on the internal bisector of ∠**B**; and I, I_3 on the internal bisector of ∠**C** (See § 135, II).

Note 2. The inscribed circle and the escribed circles are often called the **in-circle** and the **ex-circles**[1]; their centres, **in-centre** and **ex-centres**; and their radii, **in-radius** and **ex-radii** respectively.

Note 3. In § § 158, 159 are solve the problem of constructing circles to touch three given lines forming a triangle. There are four such circles.

EXERCISE LXV.

1. Can a circle be drawn to touch three given parallel straight lines?

2. Construct a triangle of sides 2 in., 2·5 in. and 3 in., and draw the inscribed circle and the three escribed circles.

3. In Fig. 382, prove that $AY_1 = AZ_1 = s$, $BX_1 = s - c$, $CX_1 = s - b$.

4. If a, b, c are the lengths of the sides **BC**, **CA**, **AB**, $2s$ the perimeter, **S** the area of the △**ABC**, and r_1, r_2, r_3 the radii of the escribed circles lying within the ∠s **A**, **B**, **C** respectively, prove that

$r_1 = S/(s-a)$, $r_2 = S/(s-b)$, $r_3 = S/(s-c)$.

5. Prove that **ABC** is the pedal[2] triangle of the triangle $I_1I_2I_3$ (Fig. 382). [AI_1 is the internal bisector of ∠**A**, and I_2I_3 is the external bisector of the same angle, ∴ I_1 **A** is ⊥ to I_2I_3].

6. Prove that **I** is the orthocentre of the triangle $I_1I_2I_3$. [See Note, § 135, IV].

7. In Fig. 381, prove that $\angle BIC = 90° + \frac{A}{2}$, $\angle CIA = 90° + \frac{B}{2}$ $\angle AIB = 90° + \frac{C}{2}$.

8. In Fig 382, prove that $\angle BI_1C = 90° - \frac{A}{2}$, $\angle CI_1A = \frac{B}{2}$, $\angle AI_1B = \frac{C}{2}$.

---

1. The escribed circles are also referred to by some writers as **e-circles**, and their radii as **e-radii**.

2. If **ABC** be a triangle, and **AD**, **BE**, **CF** are ⊥ to **BC**, **CA**, **AB** (see Fig. 399), then the △**DEF** is called the pedal triangle of the △**ABC**.

9. **A, B, C** are three given points (not in a straight line). With centres **A, B, C** describe three circles which shall touch one another externally [see Note 3, § 158]

10. In Ex. 9. construct ⊙s with centres **A, B, C**, such that the ⊙s with centres **B, C** may touch *externally*, and the ⊙ with centre **A** may touch each of the other circles *internally*.

11. Given the three ex-centres of a triangle, construct the triangle. [See Ex. 4].

12. Given a side **BC** of a triangle in position, and the magnitude of the opposite angle **A**, find the locus of the in-centre **I** of the triangle. [The angle which **BC** subtends at the in-centre is equal to $90° + \frac{A}{2}$ (see Ex. 7)]

13. In Ex. 12, find the locus of the ex-centre I_1 of the triangle **ABC**.

14. **AP, AQ** are two fixed tangents to a given circle. A variable tangent to the circle cuts **AP, AQ** in **X** and **Y**. Show that either **AX+AY+XY** or **AX+AY−XY** has a constant value. [See Ex. 2, also Note 1 § 158].

PROBLEM 11.

160. On a given straight line AB to construct a segment of a circle which shall contain an angle equal to a given angle P.

At A make ∠BAT= ∠P. Draw AO ⊥ to AT; draw LO the perpendicular bisector of AB, meeting AO in O.

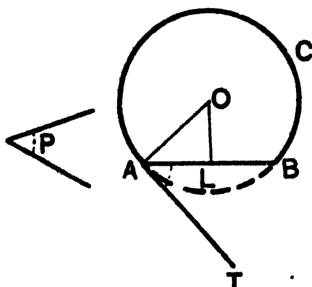


Fig. 383

Since O lies on the perpendicular bisector of AB, it is equidistant from A and B ;

∴ a circle described with centre **O** and radius **OA** will pass through **A**, **B**.

Draw this circle. Then the segment **ACB** of this circle, alternate to $\angle$**BAT**, is the required segment.

Proof. Since **AT** is ⊥ to **OA**, (Constr.)

∴ **AT** is a tangent to the circle ;

and **AB** is a chord through **A**,

∴ $\angle$**BAT** = angle in the alternate segment **ACB**.

∴ the angle in the segment **ACB** = $\angle$**P** (∵ $\angle$**BAT** = $\angle$**P** by construction).

Note. For another method see corollary, § 145.

EXERCISE LXVI.

1. On a straight line of length 5 cm. construct a segment of a circle to contain an angle of 80°.

2. Construct the locus of points at which a given line subtends an angle of 30°.

3. Draw a chord of a circle dividing it into two segments, so that the angle in one of the segments may be double the angle in the other.

4. Show how to construct on a given base an isosceles triangle with a given vertical angle.

5. Show how to inscribe a triangle **ABC** in a given circle so that $\angle$**A** is equal to a given angle, and the sides **AB**, **AC** pass through two given points **P**, **Q**, respectively.

6. **BC** is a given straight line; construct a △ **ABC** on **BC**, so that $\angle$**A** may be of a given magnitude, and the median **AD** may be of a given length. When does the construction fail?

7. Construct a triangle having given a side, the opposite angle, and the distance of the opposite vertex from this side.

161. In a given circle to inscribe a triangle equiangular with a given triangle ABC. [Euclid IV. 2]

Take any point **P** on the ⊙, and draw the tangent **XPY**.

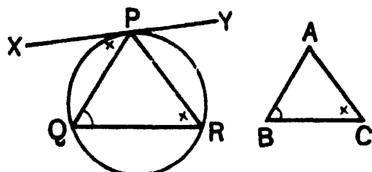


Fig. 384

Draw chords **PR, PQ** making ∠s **YPR, XPQ** respectively equal to ∠s **B** and **C**. Join **QR**. Then **PQR** is the required triangle. [Prove that ∠**Q** = ∠**B**, ∠**R** = ∠**C**, and hence ∠**QPR** = ∠**A**].

162. About a given circle to circumscribe a triangle equiangular with a given triangle ABC. [Euclid IV. 3]

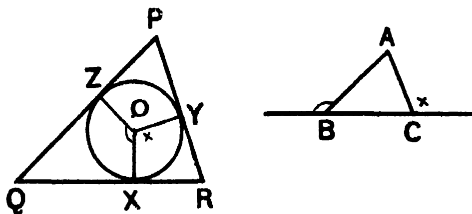


Fig. 385

Let **O** be the centre of the ⊙. Take any point **X** on the ⊙. Join **OX**.

Draw radii **OY, OZ** making ∠s **XOY, XOZ** respectively equal to the supplements of ∠s **C** and **B.**

Draw tangents **QXR, RYP, PZQ** at **X, Y, Z.**

Then PQR is the required triangle. [In the quadrilateral OXQZ, ∠s OXQ, OZQ are rt. ∠s, ∴ ∠ Q is supplement of ∠XOZ, and hence equal to ∠B ; similarly ∠R = ∠C ; ∴ ∠P = ∠A.]

Ex. 1. **Inscribe an equilateral triangle in a given circle.**

Ex. 2. **Circumscribe an equilateral triangle about a given circle.**

Ex. 3. Draw a circle of radius 2 inches. Inscribe in it, and circumscribe about it triangles whose angles are 40°, 60°, 80°.

Ex. 4. In a given circle inscribe a triangle having its sides parallel to three given straight lines.

CONSTRUCTION OF CIRCLES SATISFYING SPECIFIED CONDITIONS.

163. The method of the intersection of loci (§ 134) is to be generally followed in finding the centre of the required circle. For this purpose we recapitulate here some of the results about loci which have already been noticed in the preceding exercises.

(i) *The locus of the centres of circles of a given radius which pass through a given point is a circle of the given radius with the given point for its centre.*

(ii) *The locus of the centres of circles passing through two given points is the perpendicular bisector of the line joining the points.*

(iii) *The locus of the centres of circles touching a given line at a given point is the straight line perpendicular to the given line through the given point.*

(iv) *The locus of the centres of circles of a given radius touching a given straight line is the pair of lines parallel to the given line, lying on opposite sides of it, and at a distance from it equat to the given radius.*

(v) *The locus of the centres of circles touching two given intersecting straight lines is the pair of lines bisecting the angles between them.* (*See Theor.* 28 § 133).

(vi) *The locus of the centres of circles touching a given circle of centre* O *at a given point* A *on it, is the straight line through* O *and* A.

(vii) *The locus of the centres of circles of a given radius, which touch a given circle externally is a concentric circle.*

(viii) *The locus of the centres of circles of a given radius which touch a given circle internally is a concentric circle.*

EXERCISE LXVII.

1. Construct a circle to touch a given straight line PQ at a given point B in it, and to pass through a given point C outside the line.

2. Construct a circle to touch two given parallel straight lines and pass through a given point *between* those lines. How many such circles are possible ?

3. Construct a circle of a given radius to pass through two given points.

4. Construct a circle to pass through two given points and have its centre on a given straight line

5 Construct a circle of a given radius to touch two intersecting lines How many such circles are possible ?

6. Construct a circle of a given radius to pass through a given point and touch a given straight line When does the problem fail ?

7. Construct a circle to touch two given intersecting straight lines and have its centre on a third given straight line How many such circles are possible ?

8. Construct a circle to touch a given circle at a given point **A** and pass through a given point **B**, not on the circle

9. Construct a circle of a given radius to touch a given circle and pass through a given point not on the circle Discuss all possible cases.

10. Construct a circle to touch a given circle at a given point and have its centre on a given straight line.

11 Construct a circle of a given radius to touch a given circle and have its centre on a given line How many such circles are possible ? When does the problem fail ?

12. Construct a circle to touch a given circle and have its centre at a given point How many such circles are possible ?

13 Construct a circle of a given radius to touch a given circle and a given straight line Discuss all possible cases.

14 Construct a circle to touch a given straight line **PQ** at a given point **A**, and to touch another given straight line **RS** not parallel to **PQ**

15. Construct a circle to touch two given parallel straight lines and a third given straight line which cuts them How many such circles are possible ? Are they equal ?

16 Construct a circle to touch three given straight lines, no two of which are parallel, and which do not all meet at a point. How many solutions are possible ? [See Note 3, § 159].

17. *Construct a circle to touch a given circle (centre* O*), and a given straight line* PQ *at a given point* A. [Through A (Fig. 386) draw XAY ⊥ to PQ, and cut off AB, AC each = the radius of the given ⊙. Join OB, OC, and make ∠s BOD, COE equal to ∠s OBY, and OCY respectively. Let OD, OE meet XAY in D and E, and the given ⊙ in G and H. It is easily proved that DG = DA, and EH = EA. Hence the ⊙s with centres D, E and radii DA and EA respectively will each touch PQ at A, and the given circle at G, H respectively, (see Cor. 2, Theor. 42)].

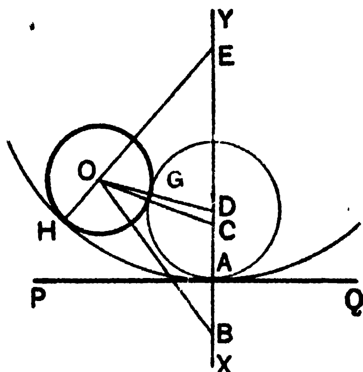


Fig 386

18. *Construct a circle to touch a given straight line* PQ, *and a given circle (centre* O*) at a given point* A.

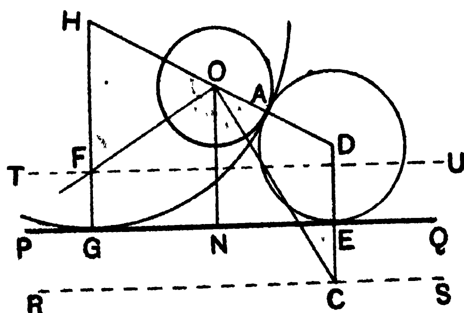


Fig. 387

[Draw RS, TU (Fig. 387) ∥ to PQ, on opposite sides of it, so that the distance of each from PQ = the radius of the given circle. Draw ON ⊥ to PQ; join OA and produce it both ways.

Bisect ∠**NOA** by **OC** meeting **RS** in **C**. Draw **CD** ∥ to **NO**, meeting **PQ** in **E** and **OA** in **D**. It is easily proved that **DE=AD.** Hence the ⊙ described with centre **D** and radius **DE** will touch the given circle at **A**, and the given line **PQ**

In the same way if the external bisector of ∠**NOA** cuts **TU** in **F**, and through **F** a parallel to **ON** is drawn cutting **PQ** in **G** and **AO** in **H**, it may be proved that **HA=HG**, so that the ⊙ with centre **H** and radius **HG** will also satisfy the given conditions.

Alternative methods for Exs. 17, 18.

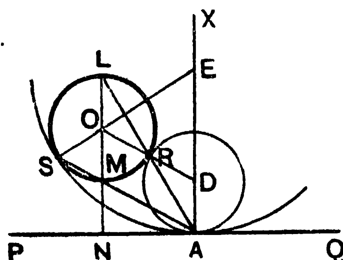


Fig. 388

[*For Ex.* 17. Draw **AX, ON** ⊥ to **PQ**, and let **ON** cut the ⊙ in **L, M.** Join **AL, AM**, and let them cut the ⊙ again in **R, S.** Join **OR, SO** and produce them to meet **AX** in **D** and **E**. It is easily proved that **DA=DR,** and **EA=ES**, so that the required ⊙s have **D, E** for their centres and **DA, EA** for radii respectively.]

[*For Ex.* 18. Draw **ON** ⊥ to **PQ**, and let it cut the ⊙ in **L, M**. Join **LA, AM** and let them cut **PQ** in **E, G**. Draw **ED, GH**, ⊥ to **PQ** to meet **OA** in **D** and **H**. It is easily proved that **DE=DA,** and **HG=HA**, so that the required ⊙s have **D, H** for centres, and **DA, HA** for radii respectively].

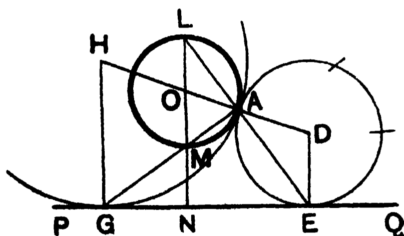


Fig. 389

There is another interesting method for Ex. 18. Draw the tangent to the given circle at **A** to cut **PQ** in **S**. Draw the internal and external bisectors of the ∠**ASQ**, cutting **OA** in **D** and **H**. Then **D** and **H** are the centres of the required circles. (see Theor. 28)

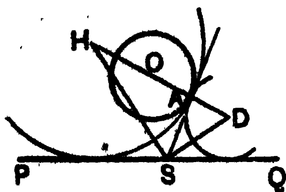


Fig. 390

19. Construct a circle to touch three given equal circles (i) so as to enclose them all, (ii) so as to enclose none of them, neither be enclosed by any of them. [The point which is equidistant from the centres of the given ⊙s, is the centre of the required circles].

20. Inscribe a circle of a given radius in a given semi-circle. When does the problem fail? [See (iv), (viii), § 163, and Ex. 13].

21 Construct a circle of a given radius to touch two given circles externally. [The dark circles are the given circles (centres **A**, **B**). Let a, b be their radii and r the given radius. With centres **A**, **B** and radii $r+a$, $r+b$ respectively describe two ⊙s (dotted) intersecting in **P** and **Q**. Then **P** and **Q** are the centres of the required circles. The proof is left to the student]. When does the problem fail?

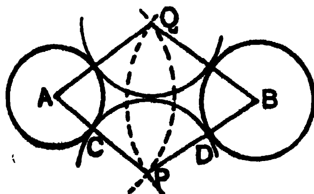


Fig. 391

22. Construct a circle of a given radius to touch each of two given circles internally. How many solutions are possible? When does the problem fail?

CHAPTER XIV.

MISCELLANEOUS.

REGULAR POLYGONS.

164. **In a given circle to inscribe a regular polygon of a given number of sides.** Before taking up the problem we shall consider the following two theorems.

1. **If the circumference of a circle is divided into a number of equal arcs, the points of division are the vertices of a regular polygon.**

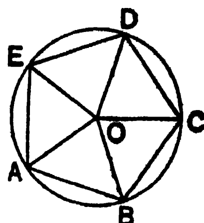


Fig. 392

Suppose that the circumference of the ⊙ ABC whose centre is O, is divided into, say, 5 equal arcs AB, BC, CD, DE, EA (Fig. 392). Then the figure ABCDE is a *regular* pentagon.

Proof. Since the arcs AB, BC,......are equal, the chords AB, BC, CD, DE, EA are equal. [*Theor.* 39 (ii)]

∴ the figure ABCDE is *equilateral.*

Again, the arcs BCDE, CDEA, DEAB, EABC, ABCD are equal, each being made up of three of the five equal arcs AB, BC, CD,...

∴ the angles which these arcs subtend at the circumference are equal (see Cor. 1, § 148), that is, ∠s BAE, CBA, DCB, EDC, AED are equal, i.e., the figure ABCDE is *equiangular.*

The figure **ABCDE** being both equilateral and equiangular is a *regular* pentagon (see § 40).

Note. By joining **OA, OB, OC, OD, OE,** it is easily seen that the isosceles △s **AOB, BOC, COD, DOE, EOA** are congruent with one another, whence also it follows that the ∠s **EAB, ABC, BCD, CDE, DEA** are equal to one another; **OA, OB, OC, OD, OE** are evidently the bisectors of these angles.

II. If a number of straight lines are drawn from the centre of a circle dividing the whole angle of 4 right angles round the centre, into as many equal parts, then the points in which these lines cut the circumference are the vertices of a regular polygon.

Suppose that OA, OB, OC, OD, OE are five lines drawn from the centre O (Fig. 392) such that ∠s AOB, BOC, COD, DOE, EOA are equal to one another, each being $\frac{360^\circ}{5}$ i.e. 72°. Then the figure ABCDE is a regular pentagon.

Proof. Since the ∠s AOB, BOC, COD, DOE, EOA are equal,

∴ the arcs AB, BC, CD, DE, EA are equal.

[*Theor.* 38 (i)]

Hence ABCDE is a regular pentagon by (I).

We are now in a position to take up the problem of inscribing in a given circle, a regular polygon of a given number of sides.

Problem. To inscribe a regular hexagon in a given circle. [Euclid IV. 15].

Let **O** be the centre of the circle. Place a chord **AB** = the radius. [See Ex. 1 (LVII)].

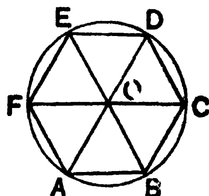


Fig. 393

Join **OA, OB.** Then the △**AOB** is equilateral.

∴ ∠**AOB** = 60°.

Place chords **BC, CD, DE, EF** each equal to the radius; each of these chords also subtends 60° at the centre.

Now ∠ **AOF** = 360° − (∠**AOB** + ∠**BOC** + ∠**COD** + ∠**DOE** + ∠**EOF**).

= 360° − 5 times 60°

= 60°

Thus the lines **OA, OB, OC, OD, OE, OF** divide the angle of 360° round the point **O**, into 6 equal parts.

∴ **ABCDEF** is a regular hexagon. [It should be noted that the sides of a regular hexagon inscribed in a circle are each equal to the radius of the circle.]

Note. Theorem II given above suggests a general method for inscribing a regular polygon of n sides in a given circle. All that is required is to draw n lines from the centre, dividing the angle of 360° round the centre into n equal parts. For this purpose the value of each part *i.e.* $\frac{360°}{n}$, is first obtained by calculation, and then n lines are drawn from the centre successively, making angles of $\frac{360°}{n}$. The points where these lines cut the circle are the vertices of the required polygon.

In all cases the lines from the centre can be laid out with the aid of a protractor. —— in some cases (*e.g.* an equilateral triangle, a square, or a regular hexagon) the lines can also be drawn with the aid of ruler and compasses only. It will be shown later (§ § 198, 199) that a regular pentagon, and a regular quindecagon can also be inscribed in a circle, with the aid of ruler and compasses only. Once a regular polygon of any number of sides has been inscribed in a circle, a regular polygon of double the number of sides can at once be inscribed by bisecting the angles which the sides of the original polygon subtend at the centre of the circle. Thus with the aid of ruler and compasses only (*i.e.* by Euclid's method) we can inscribe in a given circle, regular polygons with 3.2^n, 4.2^n, 5.2^n, 15.2^n sides where n is zero or any positive integer. It should be noted that a figure obtained with the aid of a protractor cannot be expected to be as accurate as one obtained with the aid of ruler and compasses only.

EXERCISE LXVIII

N. B. You are not to use a protractor in doing Exs. 1—4.

1. **Draw any circle and inscribe a square in it.** [Euclid IV. 6]

Draw two diameters **AB, CD,** at right angles to each other. Then **A, B, C, D** are the vertices of a required square The proof is left to the student.

2. **Draw any circle and inscribe an equilateral triangle in it.** [△ **ACE** in Fig. 393 is an equilateral △ inscribed in the ⊙. You may also do the problem by constructing an equilateral △, and then inscribing in the given circle a triangle equiangular with it by † 161 ; see Ex. 1, p. 397].

3. Draw a circle of radius 1·5 inches, and inscribe a square in it, and thence obtain a regular octagon inscribed in the same circle. Measure the perimeters of the square and the octagon.

4. Draw a circle of radius 2 inches, and inscribe a regular hexagon in it, and obtain therefrom (i) a regular do-decagon (ii) a regular polygon of 24 sides, inscribed in the same circle. Measure their perimeters.

5. Draw three circles each of radius 2 inches, and inscribe in them three regular polygons of 5, 7, and 9 sides respectively. [Use a protractor]. Measure the perimeter of each of the polygons [A regular pentagon can be inscribed in a circle, without the use of a protractor (§ 198).]

6. Inscribe regular polygons of 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, sides in seperate circles each of radius 6 cm. Measure in each case the perimeter of the polygon and calculate the ratio of the perimeter to the diameter of the circle. Arrange the results in a tabular form as shown below.

Diameter of the ⊙ = 12 cm.

Number of sides.	The perimeter in centimetres.	Ratio of the perimeter to diameter.
3		
4		
5		
6		
7		
8		
9		
10		
12		
16		

165. To circumscribe a regular polygon of a given number of sides about a given circle.

Before taking up the problem we shall consider the following theorem :

If the circumference of a circle is divided into a number of equal arcs, the tangents at the points of division form a regular polygon.

O is the centre of the ⊙, and suppose the circumference is divided into a number of equal arcs, say *five*, at the points **A, B, C, D, E.** The tangents at these points form a polygon **PQRST.** To show that **PQRST** is a *regular* polygon.

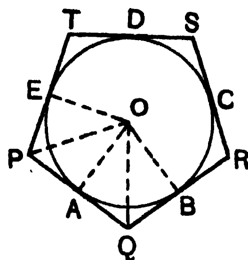


Fig. 394

Join **OE, OP, OA, OQ, OB.**

Proof. Since **PE, PA** are tangents from **P,**

∴ **PE = PA,** and **OP** bisects each of the ∠s **EOA, EPA,** (see Theor. 41 and Corollary).

Similarly, **OQ** bisects each of the ∠s **AOB, AQB.**

Since arc **EA** = arc **AB,** ∴ ∠**EOA** = ∠**AOB,**

∴ ∠**AOP** = ∠**AOQ.**

Now in the △s **AOP, AOQ, AO** is common, ∠**AOP** = ∠**AOQ** (proved), and ∠**OAP** = ∠**OAQ** being right ∠s, (Theor 40);

∴ the △s are congruent;

∴ **AP = AQ,** and ∠**OPA** = ∠**OQA.**

Thus **PQ = 2AP,** similarly it may be shown that **PT = 2EP.**

But **PA = PE,** (proved), ∴ **PQ = PT.**

Similarly it may be shown that any two consecutive sides are equal. Hence **PQRST** is equilateral.

Again ∠**EPA** = 2∠**OPA**, and ∠**AQB** = 2∠**OQA**,
∴ ∠**EPA** = ∠**AQB** [∵ ∠s **OPA**, **OQA** have already been proved to be equal]

Similarly it may be shown that any two consecutive angles are equal. Hence **PQRST** is equiangular.

It is thus proved that **PQRST** is a regular polygon.

Corollary. *The tangents drawn at the vertices of a regular polygon inscribed in a circle, form a regular polygon of the same number of sides, circumscribed about the circle.*

The above theorem and corollary give us a **method of circumscribing a regular polygon of a given number of sides, say *n*, about a given circle.** From the centre of the ⨀, draw *n* straight lines dividing the angle of 360° round the centre, into *n* equal parts (by using a protractor where necessary). Draw tangents at the points where these lines meet the circumference. The polygon formed by these tangents is the required polygon. In this connection read the remarks on page 406.

166. A circle may be inscribed in, and a circle may be circumscribed about, any regular polygon.

Let us take for example a regular polygon **ABCDE** of 5 sides.

Let the bisectors of ∠s **EAB**, **ABC** meet in **O**. Join **OC, OD, OE**.

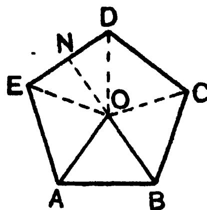


Fig. 395

In the △s **AOB**, **BOC**, **AB** = **BC**, **OB** is common, and ∠ **OBA** = ∠ **OBC**,
∴ the △s are congruent;

∴ ∠**OCB** = ∠**OAB** = ½∠**EAB** = ½∠**BCD**;
∴ **OC** bisects ∠**BCD**.

In a similar way it may be shown that **DO**, **EO** bisect ∠s at **D** and **E**.

Thus the bisectors of the ∠s of a regular polygon meet in a point.

Again ∠OAB = ∠OBA, ∴ OA = OB,

Similarly OB = OC, OC = OD, and so on.

Thus *the point* O *in which the bisectors of the* ∠s *of the polygon meet, is equidistant from the vertices of the polygon.* Hence the circle described with centre O and radius OA will circumscribe the polygon.

Again since O lies on the bisectors of the angles of the polygon, it is equidistant from the sides of the polygon (Theor. 28). Hence if perpendiculars are drawn from O to AB, BC, CD, DE, EA, these perpendiculars will be equal. Therefore the circle described with centre O, and radius = one of these perpendiculars (ON) will touch the sides of the polygon at the feet of these perpendiculars. This circle will be the inscribed circle of the polygon.

EXERCISE LXIX.

1. **To circumscribe an equilateral triangle about a given circle.** (see Cor. § 165, and Ex 2, p. 406, also § 162).

2. **To circumscribe a square about a given circle.**

3. Draw a circle of radius 1 inch and circumscribe a regular hexagon about it and measure a side.

4. Inscribe a circle in a given square.

5. Inscribe a circle in a given rhombus. [The centre* of a rhombus is equidistant from the sides].

6. Draw a circle of radius 2 inches, and circumscribe (i) a regular pentagon, (ii) a regular heptagon about it. [Use a protractor, draw separate figures.]

7. Construct a regular heptagon of side 1 inch, and inscribe a circle in it, and circumscribe a circle about it. Measure the radii of the two circles.

---

* **The intersection of the diagonals.**

8. *Show that the sum of the perpendiculars drawn to the sides of a regular* polygon from any point* O *within it, is constant* [The area of the polygon = sum of the areas of the △s into which it is divided by straight lines joining O to the vertices of the polygon Cf Ex 14 (XXXI)].

9 *Prove that the area of a square circumscribed about a circle is twice the area of a square inscribed in the same circle.*

10. An equilateral triangle is inscribed in a circle, and another equilateral triangle is circumscribed about it, show that a side of the first triangle is half a side of the second

AREA OF A CIRCLE

167. We have already seen (§ 35) that the ratio of the length of the circumference of a circle to the diameter is the same for all circles, and the value of this ratio is very nearly $\frac{22}{7}$. This ratio is usually represented by the Greek letter π, and we have:

circumference $= \pi$ times the *diameter* or 2π times the *radius.*

Denoting the radius by r, we may write,

circumference $= 2\pi r$.

The exact value of this ratio (i.e. π) cannot be found as a terminating or recurring decimal. $\frac{22}{7}$ is an approximate value of π, a more approximate value is 3·1416 (see foot-note p. 71).

1. **The area of a circle.** Take a number of points (say five), A, B, C, D, E on the circumference and draw tangents at these points. These tangents form a polygon circumscribed about the circle. If O (the centre of the ⊙) is joined to the

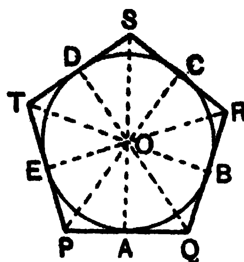


Fig 396

* The result holds if the polygon is convex and *equilateral.*

vertices of the polygon, it is divided into a number of triangles having O for a common vertex, and the sides of the polygon for opposite sides. If O is joined to A, B, C, D, E, then OA, OB, OC, OD, OE are ⊥ to PQ, QR, RS, ST, TP respectively (Theor. 40). If r be the radius of the circle,

$$\triangle \text{OPQ} = \tfrac{1}{2}\text{r. PQ}, \quad \triangle \text{OQR} = \tfrac{1}{2}\text{r. QR}, \text{ and so on.}$$

∴ the area of the polygon

$$= \tfrac{1}{2}\text{r. PQ} + \tfrac{1}{2}\text{r. QR} + \tfrac{1}{2}\text{r. RS} + \tfrac{1}{2}\text{r. ST} + \tfrac{1}{2}\text{r. TP}$$

$$= \tfrac{1}{2}\text{r. (PQ} + \text{QR} + \text{RS} + \text{ST} + \text{TP)}$$

$$= \tfrac{1}{2}\text{r} \times \text{perimeter of the polygon.}$$

This result holds for *all* polygons circumscribing the circle.

Now if an indefinitely large number of points A,B,C,...are taken on the circumference, very *close* to one another, and tangents are drawn at those points, the circumscribed polygon thus obtained, very nearly coincides with the circle, so that ultimately the perimeter of the polygon is equal to the circumference of the circle, and the area of the polygon is equal to the area of the circle. Hence we have,

area of the circle $= \tfrac{1}{2}$ radius × circumference.

$$= \tfrac{1}{2}\text{r} \times (2\pi\text{r})$$

$$= \pi\text{r}^2.$$

II. **Area of a sector of a circle.** If two sectors of the same circle have equal angles*, they are congruent, and hence their areas are equal. If the angle of one of the sectors is *m* times the angle of the other,

* The angle of a sector means the angle included between the two bounding radii.

the area of the first sector is m times the area o the second. We may say more generally that the areas of two sectors of the same circle are proportional to their angles. The circle itself may be regarded as a sector of angle 360°. Hence the **area of a sector of a circle** $= \frac{D^\circ}{360^\circ}$ of the area of the circle $= \frac{D}{360}(\pi r^2)$, where D° is the angle of the sector, and r is the radius of the circle. If θ radians be the circular measure of the angle of the sector, we have $\theta = \frac{D}{180}\pi$, (see § 36), and we have

area of the sector $= \frac{1}{2}\theta r^2$.

The student is advised to obtain directly (as in I) the area of a sector in the interesting form,

area of a sector $= \frac{1}{2}$ **radius** × **the length of the arc.**

III. Area of a segment of a circle.

Area of the segment BDC = the area of the sector **ABDC** – the area of the △ **ABC** (Fig. 397).

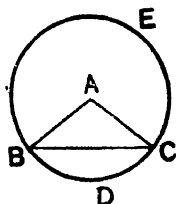


Fig. 397

EXERCISE LXX.

N. B. In the following examples it is understood that measurements of lengths or angles are to be taken (with ruler and protractor) wherever required. Take $\pi = \frac{22}{7}$.

1. Calculate the area of a circle of radius 3 inches.

2. Find the area in acres of a circular field of radius 300 yards.

3. Which is the greater in area—a circular field of radius 200 yards, or a square field of side 300 yards.

4. Find the ratio of the area of a circle to that of the inscribed square.

5. Find the ratio of the area of a circle to that of the circumscribed square.

6. The area of a circular field is 10 acres. Find the radius. [Express the answer to the nearest yard].

7. Find by calculation, to the nearest yard, the radius of a circle equal in area to a square of side 200 yards.

8. The radius of one circle is twice the radius of another, compare their areas.

9. The radius of one circle is m times the radius of another, compare their areas.

10. Find the area of the surface bounded by two concentric circles of radii 1·5 in. and 2·5 in.

11. The perimeter of a circular field is 2 miles. Find the area of the field in acres

12 *Prove that the area of the circle described on the hypotenuse of a right-angled triangle as diameter is equal to the sum of the areas of the circles described on the sides (containing the right angle) of the triangle as diameters*

13. Find the area of a sector of angle 50° in a circle of radius 3 inches.

14. Find the area of a sector in a circle of radius 4 cm, the length of the arc being 9 cm.

15. Calculate the area of a sector whose chord is 3 inches, in a circle of radius 2 inches [Find the angle of the sector by measurement.]

16. Find the area of a segment of a circle of radius 2 in, the arc of the segment subtending an angle of 60° at the centre of the circle.

17. Find the areas of the two segments into which a circle of radius 5 cm. is divided by a chord of length 8 cm.

18. Find the area of a segment of base 6 cm. and height 2 cm.

19. Find the area of a segment of base 2 cm. and height 6 cm.

20. Find the area of the portion of surface common to two circles of radii 4 cm. and 5 cm with their centres 7 cm. apart.

ORTHO-CENTRE AND PEDAL TRIANGLE.

168. We have already proved (see § 135 IV) **that the perpendiculars drawn from the vertices of a triangle to the opposite sides meet at a point.** We give another proof below.

ABC is a triangle. Draw **BE CF** ⊥ to **AC**, **AB** respectively, meeting at **P**. Join **AP** and produce it to meet **BC** in **D**.

To prove that **AD** is ⊥ to **BC**.

Join **EF**.

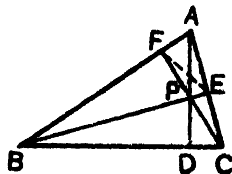


Fig. 398

Since ∠s **AEP**, **AFP** are rt. ∠s.

∴ **A**, **E**, **P**, **F** are concyclic (Theor. 37),

∠**PAF** = ∠**PEF** in the same segment.

Again, since ∠s **BEC**, **BFC** are rt. angles, **B**, **F**, **E**, **C** are concyclic, i. e., **BFEC** is a cyclic quadrilateral (Theor. 35).

∴ the ext. ∠**AEF** = int. opp. ∠**CBF**.

∴ ∠**PAF** + ∠**CBF** = ∠**PEF** + ∠**AEF** = ∠**AEP** = a rt. ∠.

Thus in the △ **ABD**, ∠**BAD** + ∠**ABD** = 1 rt. ∠.

∠**ADB** is a right angle.

i.e. **AD** is ⊥ **BC**.

Note 1. The point **P** in which the ⊥s **AD**, **BE**, **CF** meet is called the **orthocentre**, and the △ whose vertices are **D**, **E**, **F** (the feet of these ⊥s) is called the **pedal triangle**, of the △**ABC**.

If the △ **ABC** is acute-angled, the orthocentre lies within the △ (as in Fig. 398); if the △ is obtuse-angled, the orthocentre lies outside the △; and if the △ is right-angled at **C**, the point **C** is the orthocentre. Draw figures illustrating these cases. Give reasons for the above statements.

Note 2. Observe that the following sets of four points are concyclic (Fig. 398);—**A,E,P,F**; **B,D,P,F**; **C,E, P,D**; **B,C, E,F**; **C,A,F,D**; **A,B,D,E**.

169. An important property.

The orthocentre of an acute-angled triangle is the in-centre of the pedal triangle.

[Since the ∠s **BAC, CBA, ACB** are acute, **D** lies *between* **B** and **C**, **E** *between* **C** and **A**, and **F** *between* **A** and **B**, so that the ortho-centre **P** lies *within* the △ **ABC**.]

Since **C**, **D**, **P**, **E** are concyclic,

∴ ∠**PDE** = ∠**PCE**.

Since **B**, **D**, **P**, **F** are concyclic

∴ ∠**PDF** = ∠**PBF**.

But ∠s **PCE**, **PBF** are each a complement of ∠**BAC**, (∵ **CFA**, **BEA** are rt-angled △s),

∴ ∠**PCE** = ∠**PBF**;

∴ ∠**PDE** = ∠**PDF**,

that is, **PD** bisects ∠**EDF**.

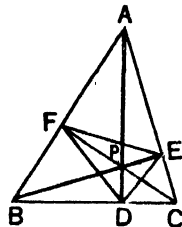


Fig. 399

Similarly **PE**, **PF** bisect ∠s **DEF**, **EFD** respectively.

Thus **AD, BE, CF bisect the angles of the pedal** △**DEF** hence **P** is the in-centre of the △**DEF** Thus the orthocentre of the original △**ABC** is the in-centre of the pedal triangle **DEF**.

EXERCISE LXXI.

1. Prove that the angles of the pedal triangle **DEF** are the supplements of 2**A**, 2**B**, 2**C**, where **A**, **B**, **C** are the angles of the original △ **ABC**, provided that the △ **ABC** is acute-angled [∠**DEF** = 2 ∠**EDP** = 2 ∠**ECP** = twice the complement of ∠**BAC** (see fig. 399)].

2. In Fig. 399, prove that **DE**, **DF** are equally inclined to **BC**; **ED**, **EF** to **CA**, and **FE**, **FD** to **AB**.

3. In Fig. 399, prove that ∠ **EDC** = ∠**A**, ∠**CED** = ∠ **B**. Prove also that each of the △s **AEF**, **BFD**, **CDE** has its angles equal to those of the △**ABC**.

4. **Prove that if ∠C of the △ ABC be obtuse, ∠EDF = 2∠A, ∠FED = 2∠B, and ∠DFE = 2∠C − 180°, where D, E, F are the feet of the ⊥s from A, B, C upon the opposite sides.**

5. **Prove that the orthocentre of an *obtuse-angled* triangle is an *ex-centre* of the pedal triangle.**

7. If I, I_1, I_2, I_3 are the in-centre and the ex-centres respectively of a triangle **ABC**, prove that of the four points I, I_1, I_2, I_3, any one is the orthocentre of the triangle whose vertices are the other three. [See Fig. 382].

8. Repeat Ex. 4, Exercise LXV.

9. Prove that **ABC** is the pedal triangle of the triangle formed by joining any three of the points I, I_1, I_2, I_3.

10. **P** *is the orthocentre of a triangle* **ABC**, *and* **AQ** *the diameter of the circumcircle through* **A**. *Prove that* **BPCQ** *is a parallelogram.* [Since **AQ** is a diameter of the circumcircle, ∴ ∠**ABQ** is a right ∠, i e. **BQ** is ⊥ to **AB**; **CP** produced is also at right ∠s to **AB**; **CP** and **BQ** are ∥, similarly **BP**, **CQ** are ∥].

11. **P is the orthocentre of a triangle ABC, and O the circum-centre. Show that AP = 2OL where L is the foot of the perpendicular from O to BC.** [**L** is the mid-point of **BC** **O** is the mid-point of the diameter **AQ** of the circumcircle, drawn through **A**. By Ex. 10, **BPCQ** is a ∥m, ∴ **PQ** and **BC** bisect each other, **L** is the mid-point of **PQ**. Now in the △**APQ**, **O** is the mid-point of **AQ**, and **L** is the mid-point of **PQ** ∴ **OL** is half of **AP**. In the same way it may be shown that **BP** = 2**OM**, **CP** = 2**ON** where **OM**, **ON** are perpendiculars from **O** upon **AC**, **AB**.]

12. **Prove that the orthocentre P, the circum-centre O, and the centroid G of any triangle ABC are collinear.**

[Let **L** be the mid-point of **BC**, and **R** the mid-point of **AP**. Join **AL**, **OP** and produce **AP** to meet **BC** in **D**. Then **OL** is ⊥ to **BC** and hence ∥ to **AD**, and = **AR** (see Ex 11). Through **L** draw a parallel to **OP** cutting **AD** in **Q**. Then **OLQP** is a ∥m and **OL** = **PQ**. Thus **AR** = **RP** = **PQ**. Through **R** draw a parallel to **OP** cutting **AL** in **H**. Let **OP** cut **AL** in **G**.

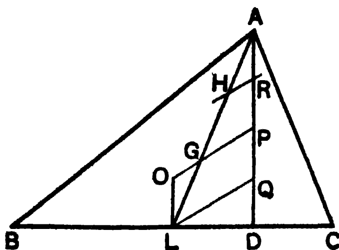


Fig. 400

It is easily proved that **AH**=**HG**=**GL** (see § 101), hence **G** is the centroid of the triangle (Note, § 135, III)

13 **ABC is a triangle; the perpendicular from A to the opposite side BC cuts BC in D and the circumcircle in Q. If P is the orthocentre show that PD=DQ.** [∠**BPD**=∠**C**=∠**BQD**, ∆s **BPD**, **BQD** are congruent (Theor 17), hence **PD**=**DQ**.]

14 **ABC** is an acute angled triangle and the perpendiculars from **A**, **B**, **C** to the opposite sides are produced to meet the circumcircle in **Q**, **R**, **S** respectively. Show that ∆ **ABC** = ∆ **BQC** + ∆ **CRA** + ∆ **ASB**

15 Construct a triangle **ABC**, given the vertex **A**, the orthocentre **P**, and the circumcentre **O**

16 Given **I**, $\mathbf{I_1}$, $\mathbf{I_2}$ [see Fig 382], construct the triangle [Cf Ex 10 (Exercise LXV).]

17 **Given a side of a triangle in position and the magnitude of the opposite angle, find the locus of the orthocentre of the triangle** [See Ex 19, Exercise LX.]

THE NINE-POINT CIRCLE

170 In any triangle the mid-points of the sides, the feet of the perpendiculars from the vertices to the opposite sides, and the mid-points of the lines joining the vertices to the orthocentre are concyclic

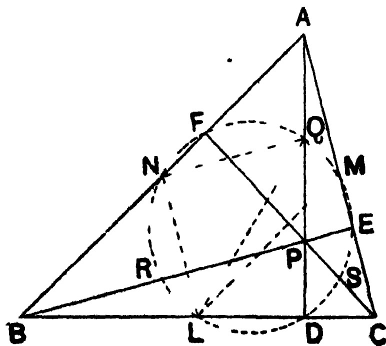


Fig. 401

ABC is a triangle **L**, **M**, **N** are the mid-points of **BC**, **CA**, **AB**, **AD**, **BE**, **CF** are ⊥ to **BC**, **CA**, **AB**. **P** is the orthocentre, **Q**, **R**, **S** are the mid-points of **AP**, **BP**, **CP**

To prove that the nine-points, **L**, **M**, **N**, **D**, **E**, **F**, **Q**, **R**, **S** are concyclic

Since **N**, **Q** are the mid-points of **AB**, **AP**, ∴ **NQ** is ∥ to **BP** i. e. to **BE**. Since **N**, **L** are the mid-points of **AB**, **BC**, ∴ **NL** is ∥ to

AC. As **BE** and **AC** are perp. to each other, ∴ **NQ**, **NL** are also perp. to each other, that is, ∠**QNL** is a right angle. Similarly ∠**QML** is a right angle. Also ∠**QDL** is a right angle because **AD** is ⊥ to **BC**.

∴ **N**, **M**, **D** lie on the ⊙ described on **QL** as diameter That is to say, **D** and **Q** lie on the circle which passes through **L**, **M**, **N**; and **QL** is a diameter of this circle. In the same way it is shown, that **E**, **R**, **F**, **S** also lie on the ⊙ which passes through **L**, **M**, **N**; and **RM**, **SN** are also diameters of this circle.

Thus the nine points **L**, **M**, **N**, **D**, **E**, **F**, **Q**, **R**, **S** are concyclic.

Note 1. The circle which passes through the nine points **L,M,N,D,E,F,Q,R,S** is called the **Nine-Point Circle** of the **ABC**. *It is evidently the circum-circle of the pedal triangle* **DEF**.

Note 2. **QL**, **RM**, **SN** are diameters of the nine-point circle and therefore bisect one another. The mid-point of **QL**, or **RM**, or **SN** is the centre of the nine-point circle. We shall denote it by the letter **U** (See Fig. 402).

Note 3. The centre of the nine-point circle is the mid-point of the straight line joining the orthocentre and the circumcentre ; i, e., U is the mid-point of OP.

Q is the mid-point of **AP**, and **L** that of **BC**. We know that **OL**$=\frac{1}{2}$ **AP**. [See Ex. 11, (LXXI)].

OL=**QP**, also **OL** and **QP** are parallel being each ⊥ to **BC**,

OQPL is a parallelogram;

∴ the diagonals **OP** and **QL** bisect each other. The point of intersection **U** is the mid-point of **OP** and **QL**, and being the mid-point of **QL**, it is the centre of the nine-point circle.* [See Note 2].

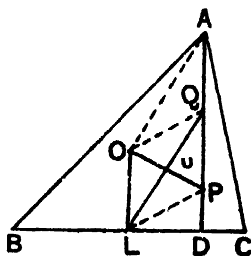


Fig. 402

* *Alternative proof.* Since **OL**, **PD** are both ⊥ to **BC** and hence parallel to each other, the perpendicular bisector of **LD** passes through the mid-point of **OP** (§ 101); similarly the perpendicular bisector of **ME** also passes through the mid-point of **OP**, where **M** is the mid-point of **CA** and **E** is the foot of the perpendicular from **B** to **CA** (see Fig. 401). But **LD**, **ME** being chords of the nine-point circle, the intersection of their perp. bisectors is the centre of that circle (Cor. 1, Theor. 29); and this point is the mid-point of **OP**.

Note 4. The ortho-centre, the centroid, the circum-centre, and the centre of the nine-point circle i. e, the points P, G, O, U, are collinear.

[The result follows at once from Note 3, and Ex 12 (LXXI)].

Note 5. The radius of the nine-point circle is half the radius of the circumcircle [Consider the $\triangle$**AOP** (Fig 402). Since **Q, U** are the mid points of **AP, OP**, (see Note 3) **QU** $= \frac{1}{2}$ **OA**. But **QU** is a radius of the nine-point circle, and **OA** is a radius of the circum-circle the radius of the nine-point circle is half the radius of the circumcircle]

Note 6. The circum-radius of the pedal triangle is half the circum-radius of the original triangle. [The result follows at once from Note 5, see Note 1]

EXERCISE LXXII

1. The circumcircle of any $\triangle$**ABC** is the nine-point circle of the triangle whose vertices are the centres of the escribed circles.

2. Prove that the $\triangle$s **BPC, CPA, APB, ABC** (Fig. 399) have the same nine-point circle [**DEF** is the pedal triangle of each of the four triangles, see Note 1]

Hence show that the circumcircles of the $\triangle$s **BPC, CPA, APB, ABC** are equal (see Ex 27, Exercise LX)

3 Prove that the $\triangle$s II_2I_3, II_3I_1, II_1I_2, $I_1I_2I_3$ have the same nine-point circle. [Fig. 382]

4 The radii of the circles circumscribing the triangle $I_1I_2I_3$, II_2I_3, II_3I_1, II_1I_2 (Fig. 382) are each double the radius of the circle circumscribing the $\triangle$**ABC**.

5. Prove that the mid-points of II_1, II_2, II_3 lie on the circum-circle of the $\triangle$**ABC**. [**I** is the ortho-centre of the $\triangle$ $I_1I_2I_3$.]

6. Prove that if two triangles have the same circumcircle and the same orthocentre, they have the same nine-point circle. [See Note 3, § 170.]

7. Given a side in position, and the magnitude of the opposite angle of a triangle, find the locus of the centre of the nine-point circle. [A circle whose centre is the mid-point of the given side].

SIMSON LINE OR PEDAL LINE.

171. If from any point P on the circumcircle of a triangle ABC, PL, PM, PN are drawn ⊥ to the sides BC, CA, AB of the triangle, then the feet of the perpendiculars, (i.e., the points, L, M, N) are collinear.

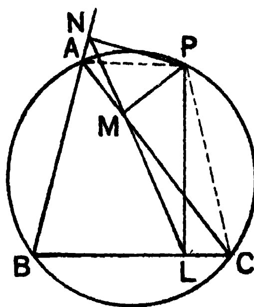


Fig. 403

Join **PA**, **PC**, **LM**, **NM**. The points **A**, **M**, **P**, **N** are concyclic [∵ ∠s **PMA**, **PNA** are right angles],

∴ ∠**PMN** = ∠**PAN**.

But **ABCP** being a cyclic quadrilateral,

∠**PAN** = ∠**PCB**; (*Theor.* 36, Corollary).

∴ ∠**PMN** = ∠**PCB**.

Again **P**, **C**, **L**, **M** are concyclic [∵ ∠**PLC** = ∠**PMC** being right angles];

∴ ∠**LMP** + ∠**PCL** = 2 rt. angles;

∴ ∠**LMP** + ∠**PMN** = 2 rt. angles.

∴ **LMN** is one straight line.

Note. The line **LMN** is called the **Pedal Line** or the **Simson Line** of the point **P** with respect to the triangle **ABC**.

EXERCISE LXXIII.

1. What are the pedal lines of the vertices of a triangle **ABC** with respect to the triangle?

2. Find the points whose pedal lines with respect to a triangle are the sides of the triangle.

3. Find the point (or points) whose pedal line with respect to a triangle **ABC** cuts the side **BC** in a given point **L**.

4. From any point **P** on the circumcircle of a triangle **ABC**, perpendiculars **PM**, **PN** are drawn to the sides **AC**, **AB** (see Fig. 403). If the line through **M**, **N** cuts **BC** in **L**, show that **PL** is ⊥ to **BC**.

5. **P** is any point such that the feet of the perpendiculars drawn from it to the sides of a given triangle **ABC** are collinear; show that the locus of **P** is the circumcircle of the △**ABC**. [Show that ∠**APC**=∠**LPN**, Fig. 403, and hence ∠**APC**+∠ **ABC**=2 rt. angles.]

6. Any point **Q** on the circumcircle of a △ **ABC** is joined to the orthocentre **P**. Show that **PQ** is bisected by the pedal line of the point **Q** with respect to the △**ABC**,

[The perpendicular **AD** from **A** to **BC** cuts the ⊙ at **E**, when produced Then **PD**=**DE**. (See Ex. 13. Exercise LXXI).

Draw **QN**, **QM** ⊥ to **BC**, **CA** respectively, then **MN** is the pedal line of **Q**; let **MN** cut **AE** in **R** and **PQ** in **T**. To prove that **PT**=**TQ**. Draw **QS** ⊥ to **AE**. Join **QC**, **QE**. We shall now prove that the △s **DRN** and **SEQ** are congruent. **QN** and **AE** being both ⊥ to **BC** are parallel, also **QS** and **BC** being both ⊥ to **AE** are parallel.

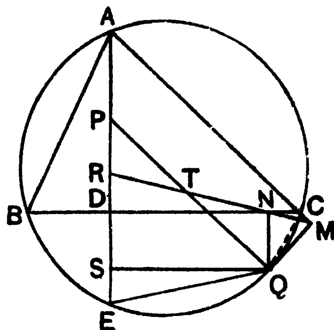


Fig. 404

∴ **QS**=**DN**, and **DS**=**QN**; also ∠**DRN**=∠**QNM**=∠**QCM**. (∵ **Q, M, C, N** are concyclic).

Again ∠**QES**=∠**QCM** (∵ **AEQC** is a cyclic quadrilateral), ∴ ∠**QES**=∠**DRN**.

Now in the △s **NRD**, **QES**, ∠**NRD**=∠**QES**, ∠**NDR**=∠**QSE** (being rt. ∠s), and **ND**=**QS**, ∴ the △s are congruent, ∴ **RD** =**ES**. But **PD**=**ED**, ∴ **PR**=**DS**=**QN**. It can now be easily proved that △s **PTR** and **QTN** are congruent so that **PT**=**QT**.

MISCELLANEOUS EXERCISE III.

1. **ABCD** is a parallelogram whose diagonals intersect in **E**. Show that the circumcircles of △s **AEB**, and **CED** touch at **E**.

2. **ABC** is an isosceles triangle inscribed in a circle The bisectors of the base angles **A** and **B**, meet the circumference in **P** and **Q**. Show that the pentagon **ABPCQ** has four of its sides equal.

How must the angles of the △**ABC** be related so that the pentagon may be equilateral?

3. Prove that if a polygon inscribed in a circle be equilateral it must be equiangular.

4. Prove that if a polygon inscribed in a circle be equiangular it must be equilateral.

5. Two circles touch externally at **P**, and a direct common tangent touches them at **A** and **B**; show that **AB** subtends a right angle at **P**

6. On the hypotenuse **BC** of a right-angled triangle **ABC**, a square is described external to the triangle. Show that the line joining **A** to the intersection of the diagonals of the square bisects ∠**BAC**.

7. **P** is the orthocentre of a △**ABC**, and the parallelogram **BPCQ** is completed. Show that **AQ** is a diameter of the circum-circle of the △ **ABC**. [See Ex. 10 Exercise **LXXI**].

8. **ABCD** is a quadrilateral in which the angles **A** and **C** are right angles. Show that the difference of the ∠s **ADB, CDB** is equal to the difference of the ∠s **CBD, ABD**. What is this difference if the acute angle between the diagonals is 50°? [The quadrilateral **ABCD** is cyclic.]

9. *If a triangle is acute-angled, its sides are the external bisectors of the angles of the pedal triangle.*

10. If a triangle is obtuse-angled, the sides containing the obtuse angle are the internal bisectors of the corresponding angles of the pedal triangle.

11. **ABC** is a triangle and **P** the orthocentre. Prove that if $\mathbf{O}_1$, $\mathbf{O}_2$, $\mathbf{O}_3$ are the circumcentres of the △s **BPC, CPA, APB,**

then $O_1P = O_2P = O_3P$ and the $\triangle O_1O_2O_3$, is congruent with the $\triangle$ **ABC** [O_1, O_2, O_3 are the intersections of the perp bisectors of **AP**, **BP**, **CP**, taken two by two See Ex 27 (LX), and Ex 2 (LXXII)]

12. Prove that the diameters of the circumscribed and the inscribed circles of a right-angled triangle are together equal to the sum of the sides containing the right angle

13 **PAQ** and **RAS** are two variable chords of a fixed circle at right angles to each other, and passing through a fixed point **A** Show that $PQ^2 + RS^2$ is constant.

14 **XAY** is a given angle, and **B** is a given point on the bisector of $\angle$**XAY** A variable circle drawn through **A** and **B** cuts **AX**, **AY** in **P** and **Q** Show that $AP + AQ =$ constant. [Chords **BP**, **QB** are equal. Draw **BD**, **BE** $\perp$ to **AX**, **AY**, and prove that $PD = QE$]

15 **AB** is a chord of a circle. Through **A**, **B** two chords **AC**, **BD** are drawn perpendicular to **AB** Prove that $AC = BD$

16. Prove that the area of a ring bounded by two concentric circles is equal to the area of a circle whose radius is equal to the tangent drawn from any point of the outer circle to the inner.

17. **ABC** is a triangle and **P** any point in **BC**. If the circum-circles of the $\triangle$s **APB**, **APC** are equal show that $AB = AC$.

18 **PQRSTU** is a hexagon inscribed in a circle, such that **PQ** is parallel to **ST**, and **QR** is parallel to **TU**. Show that **RS** is parallel to **UP**

19 A radius of one circle is a diameter of another, prove that any chord of the outer circle drawn through the point of contact is bisected by the inner circle

20 **ABCDEF** is a regular hexagon, prove that **AC** is trisected by **BD**, **BF**. [Let **BD**, **BF** cut **AC** in **P**, **Q**, prove that the $\triangle$s **BPC** and **AQB** are isosceles, and $\triangle$**BPQ** is equilateral.]

21. A straight line **AB** is trisected at the points **P**, **Q**, the point **P** being nearer to **A** than to **B** An equilateral triangle **PQR** is described on **PQ** Prove that **AR** is a tangent to the circle circumscribing the $\triangle$ **PBR**.

22. **ABC** is a triangle. The side **AB** touches the in-circle at **Z** and the ex-circle within ∠**A** at Z_1 (see Fig. 382). Prove that ZZ_1 = **BC**. [See Note 1, § 158, and Ex. 2 (LXV)].

23. **P** is a variable point on a semi-circle whose diameter is **AB**, and centre is **O**. **PQ** is drawn ⊥ to **AB**, and **OR** is the radius ⊥ to **AB**. On **OP** a point **S** is taken such that **OS**=**PQ**. Prove that the locus of **S** is a circle whose diameter is **OR**.

24. Two circles intersect in **A** and **B**. **P** is a variable point on one of them. **PA**, **PB** (produced if necessary) cut the other circle again in **Q**, **R**. Prove that the chord **QR** is of constant length and the radius of the circumcircle of △**PQR** is constant.

25. The bisector of ∠**A** of a triangle **ABC**, cuts the circumcircle of the triangle, and the bisector of ∠**C** in **D** and **E** respectively. Prove that **DE**=**DC**.

26. The four external bisectors of the angles of any quadrilateral form a cyclic quadrilateral. [Compare Ex. 23, Exercise LX]

27. **AB**, **CD** are two equal chords of a circle and **M**, **N** their mid-points. Prove that **MN** is equally inclined to the chords.

28. **ABC** is a triangle, and **P** is any point in **BC**. Prove that one of the angles between the tangents drawn at **A** to the circumcircles of the △s **APB** and **APC** is equal to ∠**BAC**.

29. Three equal circles pass through the same point **P** and intersect again, two by two, in **A**, **B**, **C**. Prove that **P** is the orthocentre of the △**ABC**. [Cf. Ex. 27 (LX). Consider in particular the case when **P** falls *within* △**ABC**. Join **AP**, **BP**, **CP** and let them meet **BC**, **CA**, **AB** in **D**, **E**, **F**. Show that ∠**PBA**=∠**PCA**, ∠**PCB**= ∠**PAB**, ∠**PAC**=∠**PBC**; whence follows, ∠**DAB**+∠**ABD**= ∠**DAC**+∠**ACD**; but∠**DAB**+∠**ABD**+∠**DAC**+∠**ACD**=∠**ABC** +∠**ACB**+∠**BAC**=2 rt. ∠s; hence ∠**DAB**+∠**ABD**=1 rt. ∠; ∴ **ADB**=a rt. angle, i.e. **AD** is ⊥ to **BC**.]

30. **AB, CD** are two fixed diameter of a circle whose centre is **O**, and **P** is a variable point on the circumference **X** and **Y** are the feet of the perpendiculars from **P** upon **AB, CD** Prove that the length of **XY** is constant for all positions of **P** (**XY** is a chord of the ⊙ with **PO** for a diameter)

31 If the circumcircle and the in-circle of a triangle have the same centre, the triangle must be equilateral

32 **O** is the circumcentre of a triangle **ABC**, and **AD** is the perpendicular from **A** to **BC** Show that ∠**BAD** = ∠**CAO**

33 **ABC** is an isosceles triangle, **AB, AC** being the equal sides A circle is drawn with a point on **AC** produced as centre, and touching **AB** at **B** **BC** is produced to meet the circle in **M.** Show that the radius through **M** is perpendicular to **AC**

34. A transverse common tangent of two circles cuts the two direct common tangents in **B** and **C** Prove that **BC** is equal in length to the portion of a direct common tangent intercepted between the points of contact [See Fig 382, and Ex 22]

35. **P** is a variable point on a given circle From **P** a straight line **PQ** is drawn parallel to a given line **AB** in the direction **AB** From **PQ** is cut off **PR** of a constant length Prove that the locus of **R** is a circle equal to the given circle

36 **QR** is a variable chord of a constant length in a fixed circle whose centre is **O**, and **S** is the mid point of **QR** From **SO** (produced if necessary) a portion **ST** is cut off equal to a given length Prove that the locus of **T** is a circle concentric with the given circle

37 **PQ** is a variable chord of a constant length in a given circle. A circle of a given radius is described so as to pass through **P, Q** Find the locus of the centre of this circle

38. Two circles intersect in **A** and **B**, and **Q** is a variable point on one of the circles **QA** and **QB** (produced if necessary) cut the other circle again in **R** and **S** Show that the locus of the circumcentre of the △**QRS** is a circle [See Exs 24, 37]

39 From the mid-point of an arc **AB** of a circle any two chords are drawn cutting the chord **AB** and the circle in **P, Q** and **R, S** respectively Prove that **P, Q, R, S** are concyclic

40. **I** is the centre of the inscribed circle and I_1, I_2, I_3, the centres of the escribed circles of a triangle **ABC**. Prove that the circle described on II_1 as diameter passes through **B** and **C**. Prove also that the circle described on the straight line joining any two of the four points **I**, I_1, I_2, I_3, as diameter, passes through two of the vertices of the △ **ABC**.

41. **I** is the in-centre of a triangle **ABC**. Prove that the circumcentres of the △s **BIC**, **CIA**, **AIB** lie on the circumcircle of the △**ABC**. [See Ex. 40 and Ex. 5 (LXXII)].

42. Through a given point outside a circle draw a straight line to cut the circle in two points **P** and **Q**, such that **PQ** may be of a given length.

43. Through a given point draw a straight line such that the perpendicular on it from another given point may be of a given length. How many such lines are possible ?

44. **P** and **Q** are two given points. Draw a circle with centre **P** such that the tangent drawn to it from **Q** may be of a given length less than **PQ**.

45. Through a given point **A** draw a straight line (or lines) to cut a given circle (centre **O**) in **P** and **Q**, such that the chord **PQ** may be bisected by a given chord **BC** of the circle. [On **OA** as diameter describe a circle cutting **BC** in **D**,**E**, then **AD**, **AE** are the required straight lines. When does the problem fail ?].

46. **B**, **C** are the extremities of a given arc (**BDC**) of a circle. Find a point **A** in the arc **BDC** such that **AB**+**AC** may be equa to a given length. (If **A** is the required point and **BA** is produced to **P** such that **AP**=**AC**, then **AB**+**AC**=**BP**, and it is easily proved that ∠**BPC**=$\frac{1}{2}$∠**BAC**. This analysis suggests the following method for the determination of the point **A**. On **BC** describe a segment **BEC** so as to contain an angle=half the angle in the segment **BDC** (see § 160) With centre **B** and radius equal to the given length describe a circle cutting the arc **BEC** in **P**. Join **PB**, and let **PB** cut the given arc **BDC** in **A**. Then **A** is the required point. The proof is left to the student. When does the problem fail ?

47. **B, C** are the extremities of a given arc (**BDC**) of a circle. Find a point **A** in the arc **BDC**, such that **AB** may exceed **AC** by a given length. (If **A** is the required point, and from **AB** is cut off **AP** = **AC**, then **AB** − **AC** = **BP**, and it is easily proved that ∠**BPC** = 90° + ½ ∠**BAC**. This analysis suggests, as in the preceding example, a method for the determination of the point **A**. When does the problem fail?

48. Two equal circles cut orthogonally. Show that the area of the portion common to the ⊙s, together with the square on the radius, is equal to half the area of either circle.

49. Construct a triangle, given a side, the opposite angle, and the sum of the other two sides. [See Ex. 46]

50. Construct a triangle, given a side, the opposite angle, and the difference of the other two sides. [See Ex. 47]

51. Construct a triangle having given one side, the sum of the other two sides and the radius of the circumcircle.

52. Construct a triangle having given one side, the difference of the other two sides and the radius of the circumcircle.

53. Construct a triangle **ABC** having given the perimeter, the angle **A**, and the length of the perpendicular from **A** to **BC**. [See Fig. 243, and note that ∠**DAE** = 90° + ½∠**BAC**.]

54. Construct a triangle having given the circum-radius and two angles. [This is the same problem as inscribing in a given circle a triangle with given angles.]

55. Construct a triangle having given the radius of the inscribed circle, and two angles. [This is the same problem as circumscribing a triangle with given angles, about a given circle.]

56. Construct a triangle given the circumcentre, the in-centre and an ex-centre. [See Ex. 5. (LXXII) and Ex. 40].

57. Construct a triangle, given the perimeter, one of the angles, and the radius of the inscribed circle. [Make an ∠**PAQ** = given angle, and draw a circle of the given radius to touch **AP**, **AQ** at **X**, **Y**. From **AP**, **AQ** cut off AX_1, AY_1

each = half the given perimeter. Describe the circle touching **AP, AQ** at X_1, (such a ⊙ is possible). Draw a transverse common tangent to the two circles described above, cutting **AP, AQ** in **B. C.** Then **ABC** is the required triangle (see Ex 2, LXV). For the triangle to be possible it is necessary that AX_1 should be so much greater than **AX** that the two ⊙s may not intersect, so that a transverse common tangent may be possible]

58. Construct a triangle, given an angle, the perpendicular distance of the vertex of this angle from the opposite side, and the radius of the inscribed circle When does the problem fail ? [See Ex. 43]

59. Construct a square such that two adjacent sides may pass through two given points **P, Q**, and the intersection of the diagonals may be at a third given point **O.** [If **A** be the vertex of the required square, in which the two adjacent sides passing through **P, Q**, meet, then ∠**PAQ** = a rt ∠, and **AO** bisects the ∠**PAQ**].

60 Calculate the length of a direct common tangent of two circles whose radii are r and r', and the distance between the centres is d, given that d is greater than $r - r'$.

61 Calculate the length of a transverse common tangent of two circles whose radii are r, r', and the distance between the centres is d, given that d is greater than $r + r'$.

62 If the radii of two circles are 5 cm. and 8 cm., and the distance between their centres is 18 cm , calculate the lengths of the direct common tangents, and the inverse common tangents.

63. Construct a rhombus so that two sides may lie along two given parallel lines, and the other two sides may pass through two given points. [Note that the perpendicular distances between the pairs of opposite sides of a rhombus are equal, see Ex 43]

64. Two circles intersect in **A** and **B.** A variable line **PAQ** is drawn through **A** cutting the ⊙s in **P, Q.** Show that **PQ** is greatest when it is parallel to the line of centres. [Let the perpendi-

culars from the centres of the ⊙s to **PQ** meet **PQ** in **S** and **T**; then **PQ** = 2 **ST**; See Ex. 2 (LVIII)].

65. Two circles intersect in **A**, **B**. Through **B** a variable line **PBQ** is drawn cutting the ⊙s in **P**,**Q**. Show that the area of the △ **APQ** is greatest when **PBQ** is parallel to the line joining the centres of the ⊙s. [Area of the △ **APQ** = $\frac{1}{2}$ **PQ** × perp. from **A** on **PQ**; when **PQ** is ∥ to the line of centres, it is of the greatest length (see Ex 64), and at the same time the perpendicular from **A** on **PQ** attains its greatest value, being then equal to **AB**].

66. Two circles intersect in **A**, **B**. Show how to draw a straight **PAQ** through **A** meeting the ⊙s in **P**, **Q**, so that **PQ** may be of a given length When does the construction fail? [See hint, Ex. 64].

67. Given an arc of a very large circle whose centre lies outside the sheet of paper on which the arc is supposed to be drawn. **A** is any point taken near the mid-point of the arc, but not *on* the arc. Show how to construct a circle with centre **A** so as to touch the given arc.

68. Describe three circles of radii a, b, c, which shall touch one another. Consider all possible cases that may arise.

69. A straight line **AB** is produced to a point **C**, and through **C** a perpendicular **CX** is drawn to **AC**. Find the point in **CX** at which **AB** subtends the greatest angle. [Describe a circle to pass through **A**, **B** and touch **CX**; the point at which this ⊙ touches **CX** is the required point. (Note that the centre of this ⊙ lies on the perp. bisector of **AB**, and radius = **CN** where **N** is the mid-point of **AB**)].

70. A flagstaff 25 feet long is erected on a tower 150 feet high. Draw a figure to a suitable scale and find the distance from the foot of the tower at which the flagstaff subtends the greatest angle. Measure this angle? [See Ex. 69.]

PART III.

RECTANGLES AND SQUARES.

CHAPTER XV.

Geometrical Theorems corresponding to certain algebraical identities.

172. **Area of a rectangle.** We have already seen (§ 111) that if the sides of a rectangle are a units of length and b units of length, the area of the rectangle is ab square units.

If AB, CD are two straight lines, any rectangle whose adjacent sides are equal to AB and CD, is called **'the rectangle contrained by AB, CD,'** and is briefly written 'rect. AB. CD'; its area is AB. CD (i.e., the product of the sides of the rectangle, see foot-note, p. 255). The notation 'AB. CD' is also used to mean "*the rectangle contained by* AB, CD." In the same way, the square on AB will mean any square whose side is equal to AB in length; and 'AB^2' may be taken to mean '*the square on* AB', it also expresses the measure of this area.

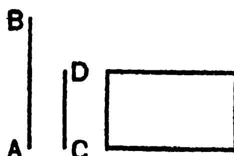


Fig. 405

The statement,

$$AB^2 = PQ.\ RS$$

may be regarded in a geometrical way as stating,

'*The square on* AB *is equal in area to the rectangle contained by* PQ, RS;

or it may be regarded in a purely algebraical way as stating,

'The square of the number of units of length in AB *is equal to the product of the numbers of units of length in* PQ, RS.

Ex. 1. Write down the values of the areas of the rectangles whose sides are

(i) a in b in., (ii) p cm, q cm, (iii) x ft, y ft, (iv) c in., c in (a square)

Ex 2 Find the areas of the rectangles whose sides are

(i) $(a+b)$ in, and c in, (ii) $(a+b+c)$ cm, d cm. (iii) $(a-b)$ in, c in., (iv) $(a+b)$ in, $(c+d)$ in (v) $(a+b)$ in, $(a-b)$ in.

Ex. 3. What is the area of a square whose side is $(a+b)$ in ?

Ex. 4 What is the area of a square whose side is $(a-b)$ in. ?

173. The geometrical theorem corresponding to, and illustrating the identity.

$$k(a+b+c+\quad)=ka+kb+kc$$

THEOREM 44. [Euclid II, 1].

If there are two straight lines, one of which is divided into any number of parts, and the other is undivided, then the rectangle contained by the two lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.

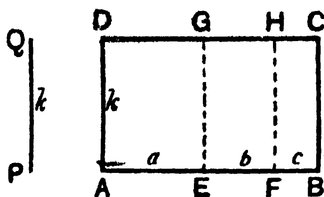


Fig. 406

AB, **PQ** are two straight lines, and **AB** is divided into any number of parts (say three), viz., **AE**, **EF**, **FB**.

Then rect. **PQ**. **AB** = rect. **PQ**. **AE** + rect. **PQ**. **EF** + rect. **PQ**. **FB**.

Proof. Draw **AD** ⊥ to **AB** and equal to **PQ** and complete the rectangle **ABCD**. Through **E**, **F** draw ||ˢ to **AD** completing the rectangles **AG**, **EH**, **FC**.

Now rect. **AC** = sum of the rectangles **AG**, **EH**, **FC**.

But **AC** = rect. **PQ**, **AB**, and **AG**, **EH**, **FC** are rectangles contained by **PQ** and **AE**, **EF**, **FB** respectively.

$$\therefore \quad PQ.AB = PQ.AE + PQ.EF + PQ.FB.$$

If **PQ** = k units of length, and **AE**, **EF**, **FB** respectively equal to a, b, c units of length (so that **AB** = $a+b+c$ units of length) we have $k(a+b+c)=ka+kb+kc$.

Note. If you were asked to illustrate geometrically the algebraical formula $k(a+b+c)=ka+kb+kc$, you would draw a straight line **AEFB** (Fig. 406) taking **AE** = a units of length, **EF** = b units, and **FB** = c units. You would next draw **AD** ⊥ to **AB** and make it equal to k units of length. On completing the rectangles **AC**, **AG**, **EH**, **FC**, and proceeding as in Theor. 44 you would at once obtain the result,

$$k(a+b+c)=ka+kb+kc.$$

EXERCISE LXXIV.

1. Show geometrically that
   (i) $(5+8)\times 3=5\times 3+8\times 3$;
   (ii) $2\times(7-4)=2\times 7-2\times 4$;
   (iii) $3(a+5)=3a+15$;
   (iv) $c(a-b)=ca-cb$, (where $a>b$).

2. **AB** is a st. line and **C** any point in it; prove that $AB^2=AB.AC+AB.CB$. [$AB^2=AB.AB=AB(AC+CB)=AB.AC+AB.CB$].

3. **AB** is produced to any point **C**; prove that $AB.AC = AB^2 + AB.BC$.

4. **AB** is divided at **C** into two parts **AC**, **CB**; prove that $AB^2 = AC^2 + 2AC.CB + CB^2$.

[$AB^2 = AB.AB = AB(AC+CB) = AB.AC + AB.CB$.

Now $AB.AC = (AC+CB)AC = AC^2 + CB.AC$. and $AB.CB = (AC+CB)CB = AC.CB + CB^2$. Hence $AB^2 = AC^2 + CB^2 + 2AC.CB$.]

174. The geometrical theorem corresponding to, and illustrating the identity,

$$(a+b)^2 = a^2 + b^2 + 2ab.$$

THEOREM 45 [II. 4].

If a straight line is divided into two parts, then the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the two parts.

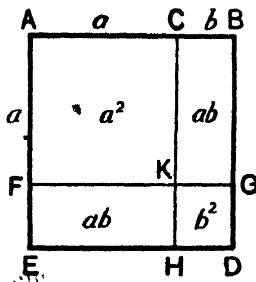


Fig. 407

The straight line **AB** is divided into two parts **AC**, **CB** at **C**. Then $AB^2 = AC^2 + CB^2 + 2AC.CB$.

Proof. On **AB** describe the square **ABDE**.

From **AE** cut off **AF** = **AC**, then **FE** = **CB**.

Draw **CH** ‖ to **AE** and **FG** ‖ to **AB**.

Let **CH**, **FG** cut in **K**.

Then **AFKC** is the square on **AC**, and **KGDH** is the square on **CB**.* Also the rectangles **CBGK** and **EFKH** are each a rectangle contained by **AC, CB** (∵ **CK** = **FK** = **AC** and **EF** = **CB**).

Now fig. **AD** = fig. **AK** + fig. **KD** + fig. **CG** + fig. **FH**.

$$\therefore AB^2 = AC^2 + CB^2 + AC.\ CB + AC.\ CB.$$
$$= AC^2 + CB^2 + 2AC.CB.$$

If **AC** = a units of length, and **CB** = b units, then **AB** = $(a+b)$ units, and we have :

$$(a+b)^2 = a^2 + b^2 + 2ab.$$

Note. If you were asked to illustrate geometrically the algebraical formula, $(a+b)^2 = a^2 + b^2 + 2ab$, you would draw a straight line **ACB**, taking **AC** = a units of length and **CB** = b units [so that **AB** = $(a+b)$ units]; then proceeding with the construction as in Fig. 407 you would prove as in Theorem 45 that

$$(a+b)^2 = a^2 + b^2 + 2ab$$

Ex. Illustrate the following by means of figures,

(i) $(5+3)^2 = 5^2 + 3^2 + 2(5+3)$;

(ii) $(a+2)^2 = a^2 + 4a + 4$;

(iii) $(b+5)^2 = b^2 + 10b + 25$.

175. The geometrical theorem corresponding to, and illustrating the identity,

$$(a-b)^2 = a^2 + b^2 - 2ab.$$

THEOREM 46. [Euclid II, 7].

The square on the difference of two straight lines is equal to the sum of the squares on the two lines

* The square on **CB**, means any square whose side is equal to **CB**.

diminished by twice the rectangle contained by the lines.

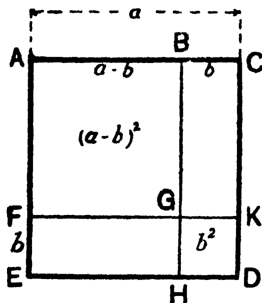


Fig. 408

AB is a straight line which is produced to **C**; evidently **AB** is the difference of the straight lines **AC, BC**.

To prove that $AB^2 = AC^2 + BC^2 - 2AC.BC$.

Proof. On **AC** describe the square **ACDE**. From **AE** cut off **AF** = **AB**. Then **FE** = **BC**. Draw **BH** ∥ to **AE** and **FK** ∥ to **AC**. Let **BH, FK** cut in **G**. Then **ABGF** is the square on **AB**, **GKDH** is the square on **BC**, and the rectangles **FD, HC** are each a rectangle contained by **AC, BC**.

Now ABGF = ACDE − BCDH − EFGH
= ACDE − BCDH − EFKD + GKDH.

$\therefore \quad AB^2 = AC^2 - AC.BC - AC.BC + BC^2.$
$= AC^2 + BC^2 - 2AC.BC.$

If **AC** = a units of length, and **BC** = b units, then **AB** = $(a-b)$ units, and we have,

$$(a-b)^2 = a^2 + b^2 - 2ab.$$

Note. If you were asked to illustrate geometrically the formula $(a-b)^2 = a^2 + b^2 - 2ab$, you would draw a straight line **ABC**, taking **AC** = a units of length, and **BC** = b units [so that **AB** = $(a-b)$

units] · then proceeding with the construction as in Fig. 408, you wou'd prove as in Theorem 46, that $(a-b)^2 = a^2 + b^2 - 2ab$.

Ex. Illustrate the following by means of figures :

(i) $(7-3)^2 = 7^2 + 3^2 - 2(7 \times 3)$;

(ii) $(a-5)^2 = a^2 - 10a + 25$. (where $a > 5$) ;

(iii) $(8-b)^2 = b^2 - 16b + 64$ (where $b < 8$).

176. The geometrical theorem corresponding to, and illustrating the identity,

$$a^2 - b^2 = (a+b)(a-b).$$

THEOREM 47. [Euclid II, 5 and 6.]

The difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.

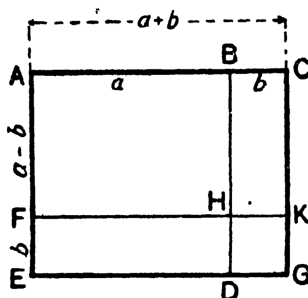


Fig. 409

Let the two lengths **AB, BC** (of which **AB** is supposed to be the greater) be placed in the same straight line.

To prove that **AB**^{2} − **BC**^{2} = (**AB** + **BC**) (**AB** − **BC**).

Proof. On **AB** describe the square **ABDE**. From **EA** cut off **EF** = **BC**. Then **FA** = **EA** − **EF** = **AB** − **BC**.

Draw CG ∥ to AE to meet ED produced in G. Draw FHK ∥ to AC.

Then DGKH is the square on BC, and ACKF is the rectangle contained by the sum and difference of AB, BC, for ACKF = AC.FA = (AB + BC) (AB − BC)

Now ACKF = ABDE + BCGD − EFKG.

But BCGD = EFHD (being each contained by AB, BC).

∴ ACKF = ABDE + EFHD − EFKG
= ABDE − DGKH
= $AB^2 - BC^2$;

thus $AB^2 - BC^2 = (AB + BC)(AB - BC)$.

If AB = a units of length, and BC = b units, we have

$$a^2 - b^2 = (a + b)(a - b).$$

Ex. Illustrate geometrically

(i) $x^2 - 4 = (x + 2)(x - 2)$ [x being > 2];
(ii) $(y + 3)(y - 3) = y^2 - 9$ [y being > 3];
(iii) $25 - z^2 = (5 + z)(5 - z)$ [z being < 5].

Corollary. If C be the mid-point of a given straight line AB, and P be any other point in the line or in the line produced, then the rectangle contained by AP, PB = the difference of the squares on half the line i.e. AC (or BC) and PC.



Fig. 410

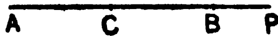


Fig. 411

(1) *When* P *is in* AB (say between C and B, Fig. 410), AP = AC + CP, PB = CB − CP. = AC − CP.

∴ AP.PB = (AC + CP) (AC − CP) = $AC^2 - CP^2$.

(ii) *When* P *is* in AB *produced* (Fig. 411),

AP = AC + CP, BP = CP − CB = CP − AC.

∴ AP. BP = (CP + AC)(CP − AC) = $CP^2 - AC^2$.

The student is advised to prove the cases when P lies between A and C, or is in BA produced.

177. **a) Illustrate geometrically the identity** $(a+b)^2-(a-b)^2=4ab$. (where $a>b$).

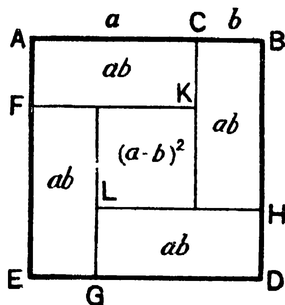


Fig. 412

Draw ACB such that AC $=a$ (units of length) CB $=b$. On AB describe the square ABDE. Cut off AF, EG, DH each equal to b. Complete the figure as shown in Fig. 412.

A glance at the figure shows that

sq. AD $=(a+b)^2$ sq. units;

LK (which is easily seen to be a square) $=(a-b)^2$ sq. units;

rect. AK $=$ rect. CH $=$ rect. DL $=$ rect. FG $=ab$.

Now sq. AD $-$ sq. KL $=$ rect. AK $+$ rect. CH $+$ rect. DL $+$ rect. FG.

i.e., $(a+b)^2-(a-b)^2=4ab$.

(*b*) **Illustrate geometrically the identity,**

$(a+b)^2+(a-b)^2=2(a^2+b^2)$ (where $a>b$).

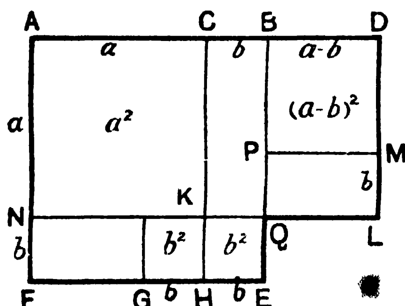


Fig. 413

Draw ACBD such that AC $=$ CD $=a$ (units of length), CB $=b$; then AB $=a+b$, and BD $=a-b$.

Describe the square ABEF. Cut off FN, EH, HG each equal to b.

Draw NL ∥ to AB, and DL ∥ to AF. Cut off LM $=b$.

Complete the figure as shown in Fig. 413.

It is seen at once that AK=CL=a^2 (sq. units.) so that AL=$2a^2$; GK=EK=b^2, so that GQ=$2b^2$:

also AE=$(a+b)^2$, BM=$(a-b)^2$; and NG=QM =$b(a-b)$.

Now AL=AQ+QM+MB=AQ+NG+MB.

∴ AL+QG=AQ+NG+MB+QG.

But AQ+NG+QG=AE.

∴ AL+QG=AE+BM.

i.e., $2a^2+2b^2=(a+b)^2+(a-b)^2$.

178. Segments of a straight line.

If a point P is taken in a straight line AB, [Fig. 414], the line AB is divided at P into two parts, AP, PB which are called **segments**; the point P is called the **point of section.**

If we take a point P in AB (or BA) produced, (Fig. 415), in that case also the line AB is said to be divided at P into two segments AP, PB. When P is *between* the points A and B as in (Fig. 414), the section is said to be **internal**; and when P is in AB (or BA) *produced*, the section is said to be **external.**



Fig. 414



Fig. 415

It should be observed that the segments are in each case the distances of the extremities (A, B) of the line from the point of section (P). In the case of internal section AB=AP+PB (*sum* of the segments), and in

the case of external section **AB=AP−BP**, (*difference* of the segments).*

EXERCISE LXXV.

1. In Fig. 410, show that if the point of section **P** approaches the mid-point **O** nearer and nearer, the rect. **AP. PB** increases more and more in area, and it becomes greatest when **P** coincides with **O**.

2. If x and y are two positive variable numbers such that $x+y$ is always equal to a constant number a, prove that the product xy is greatest when $x=y=\frac{a}{2}$

3. In Fig. 410, if $\mathbf{AP.PB=3PO^2}$, show that $\mathbf{PB=\frac{1}{4}AB}$. Find the ratio of **AP** to **PB**.

---

* Euclid regarded all lengths as *positive*: a negative length had as yet no meaning for us. But in order to widen the scope of Geometry, and in order to make further progress in the subject possible, it was found necessary to introduce the idea of *negative* lengths. The convention by means of which this has been effected in Modern Geometry is to regard all lengths measured on a line *in one direction* as *positive*, and those measured *in the opposite direction* as *negative*. Thus if the line **AB** (extending *from* **A** *to* **B**) be regarded as positive, then the line **BA** (extending *from* **B** *to* **A**) will be regarded as negative. **AB** and **BA** have the same absolute length but differ in sign. Hence we may write,

$$\mathbf{BA=-AB} \text{ or } \mathbf{AB+BA=0}.$$

When a line **AB** is divided (*internally* or *externally*) at **P**, the two segments are taken to be **AP** (extending *from* **A** *to* **P**) and **PB** (extending *from* **P** *to* **B**).

In the case of internal section. **AP, PB** are of the same sign as **AB**, and we have **AB=AP+PB**.

In the case of external section, **AP, PB** have opposite signs (being in opposite directions) and we have

$$\mathbf{AB=AP-BP=AP+PB}.$$

Thus in both cases, **AB=AP+PB**.

It should also be noted that in the case of internal section the ratio $\mathbf{\frac{AP}{PB}}$ is positive, but in the case of external section, $\mathbf{\frac{AP}{PB}}$ is negative.

These remarks are however not intended to apply to the text of this book (Parts I—III), where we are considering all lengths to be positive.

4. Illustrate geometrically the following identities :

(i) $(a+b)(c+d)=ac+ad+bc+bd$;

(ii) $(a-b)(c+d)=ac+ad-bc-bd$, $[a>b]$;

(iii) $(a-b)(c-d)=ac+bd-ad-bc$, $[a>b,\ c>d]$.

5. If AB is bisected at C, and divided internally (or externally) at P into two unequal segments AP, PB, prove that $AP^2+PB^2=2(AC^2+CP^2)$. [Cf Cor., § 176].

6. In Fig. 410, show that if the point **P** approaches **C** nearer and nearer, AP^2+PB^2 diminishes more and more, and it becomes least when **P** coincides with **C**.

7. *If x, y be two variable positive numbers such that $x+y=a$ (constant), prove that x^2+y^2 is least when $x=y=\frac{a}{2}$.*

8. In Ex. 5, prove that if $AP>PB$, then $AP^2-PB^2=2AB.CP$.

9. The sum of the squares on two straight lines can never be less than twice the rectangle contained by the lines. [The sum of the squares on the two lines exceeds twice the rectangle contained by the lines, by the square on the difference of the two lines (see Theor. 46)].

10. Prove that the square on the difference of the two sides of a right-angled triangle together with twice the rectangle contained by these sides is equal to the square on the hypotenuse. [Let the $\triangle ABC$ be right-angled at **C**; we have $(AC-BC)^2+2AC.BC=AC^2+BC^2-2AC.BC+2AC.BC=AC^2+BC^2=AB^2$. (Theor 25)].

11. **ABC is a triangle right-angled at A. AD is drawn $\perp$ to BC. Prove that $AD^2=BD.DC$.** [First prove that $2AD^2=BC^2-BD^2-CD^2$]

12. **In Ex. 11, prove that $AC^2=BC.DC$ and $AB^2=BC.BD$.** [See also note 3, p. 289].

13. *If* **P** *be a point on* **AB** *produced, prove that* $AP^2-BP^2>AB^2$.

14. **AB** *is a straight line and* **P** *any point in it, prove that the distance of the mid-point of* **AB** *from* **P** *is half the difference of the distances of* **A**, **B** *from* **P**.

15. **AB** *is a straight line and* **P** *any point on* **AB** (*or* **BA**) *produced, prove that the distance of the mid-point of* **AB** *from* **P** *is half the sum of the distances of* **A**, **B** *from* **P**.

16. A straight line **AB** is bisected at **C** and is divided internally at another point **P**. Prove that four times the square on **PC** together with twice the rectangle contained by the segments **AP**, **PB** is equal to the sum of the squares on **AP**, **BP**.

17. A straight line **AB** is bisected at **C** and produced to **P**. Prove that four times the square on **PC** *minus* twice the rectangle contained by **AP**, **PB** is equal to the sum of the squares on **AP**.**PB**.

18. If **O**, **A**, **B** be any three collinear points in order, show that $OA^2+OB^2=AB^2+2OA.OB$.

19. **A**, **B**, **C**, **D** are any four points in order, in a straight line; prove that $AD.BC=BD.AC-AB.CD$.

20 **A**, **B**, **C**, **D** are any four points in order, in a straight line; **P** is the mid-point of **AB** and **Q** is the mid-point of **CD**; prove that $2\,PQ\,AB=AC.AD-BC.BD$. [Denote the distances of **B**, **C**, **D** from **A** by b, c, d.]

21. **A**, **B**, **C**, **D** are any four points in order, in a straight line, prove that

(i) $AC^2+BD^2=AB^2+CD^2+2AD\,BC$;

(ii) $AD^2+BC^2=AC^2+BD^2+2AB.CD$.

22. **ABC is any triangle. D is the mid-point of BC and AN is perpendicular from A to BC. If** $AB>AC$, **show that** $AB^2-AC^2=2BC.DN$.

23 If from any point **O** there be drawn a perpendicular **ON** to the line **AB**, prove that $OA^2-OB^2=AB^2+2AB.BN$, if **N** lies on **AB** produced. What would be the form of the result if **N** fell between **A** and **B**?

24. **A**, **B**, **C**, **D** are four points in order on a straight line, such that $AB.CD=BC.AD$. If **O** be the mid-point of **AC**, prove that $OA^2=OB.OD$. [From the given relation, since $AD>CD$, **AB** must be > **BC**, **O** must be between **A** and **B**. Let $OA=OC=a$, $OB=b$, and $OD=d$. From the given relation we have $(a+b)(d-a)=(a-b)(a+d)$].

CHAPTER XVI.

SQUARES ON THE SIDES OF A TRIANGLE. EXTENSION OF PYTHAGORAS' THEOREM.

179. Projection. If from the extremities of a

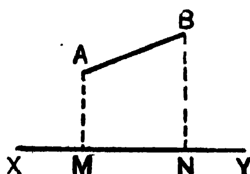


Fig. 416

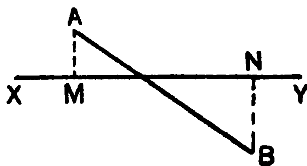


Fig. 417

line **AB** perpendiculars **AM, BN** are drawn to a straight line **XY**, then **MN** is called **the projection** of **AB** upon **XY**.

It may sometimes be necessary to produce the line **XY** upon which we project. (see Fig. 418).

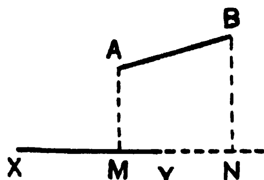


Fig. 418

Note. The projection of a point upon a straight line is the foot of the perpendicular drawn from the point to the line. Thus **M** and **N** are the projections of **A** and **B** on **XY**.

EXERCISE LXXVI.

1. Draw any triangle **ABC**, and any line **PQ**. Project the sides of the triangle upon **PQ**.

2. In Fig. 293, name the projection of **AB** upon **BC**, of **AB** upon **AC**, of **BC** upon **AB**, of **CA** upon **AB**.

3. Draw a regular hexagon **ABCDEF** of side 1 inch Project **AB**, **AC**, **AD**, **AE**, **AF** upon **AB**, also upon **AE**; and measure the projections.

4. **ABC** is a triangle, and **PQ** any straight line, show that the projection of **AC** upon **PQ** is equal either to the sum or to the difference of the projections of **AB**, **BC** upon **PQ**. Draw two figures showing the two different cases.

5. **C** is the mid-point of a straight line **AB**. If the projections of **A**, **B**, **C** upon any line **PQ** be **L**, **M**, **N**, show that **N** is the mid-point of **LM**.

6. Prove that the projection of a straight line upon any straight line can never exceed in length the line of which it is the projection.

7. Prove that if two straight lines are equal and parallel, their projections upon any straight line are equal.

8. Prove that the projections of a given straight line upon two parallel straight lines are equal.

9. **PQ** is a fixed straight line, and **AB** is a line of a given length capable of turning round the point **A**. Show how the projection of **AB** upon **PQ** varies in magnitude as **AB** turns round **A**. When does the projection become greatest and when least?

10 Under what circumstances is the projection of a line equal to the line itself?

11. Under what circumstances does the projection of a line become zero?

12. **If ∠BAC be 60°, prove that the projection of AB upon AC is $\frac{1}{2}$ AB.**

13. If x, y **be the projections of a line l upon any two lines at right angles to each other prove that $l^2 = x^2 + y^2$.**

14. A man ascends a distance of 1 mile on an incline of 10° to the horizon. Find the height of the second position above the original position. [Draw a diagram to scale.]

THEOREM. 48. [Euclid II, 12]

180. In an obtuse-angled triangle the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle plus twice the rectangle contained by one of those sides and the projection on it of the other.

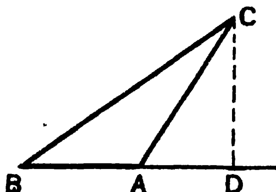


Fig. 419

ABC is a triangle obtuse-angled at A. CD is the perpendicular from C upon AB, meeting BA produced in D; then

AD is the projection of AC upon BA.

To prove that $BC^2 = AB^2 + AC^2 + 2BA.AD$.

Proof. Since △ CBD is right-angled at D,

∴ $BC^2 = BD^2 + CD^2$. (*Pythagoras' Theorem*)

Since $BD = BA + AD$,

∴ $BD^2 = BA^2 + AD^2 + 2BA.AD$; (*Theor.* 45)

∴ $BC^2 = BA^2 + AD^2 + 2BA.AD + CD^2$.

But △ ADC is right-angled at D,

∴ $AD^2 + CD^2 = AC^2$;

∴ $BC^2 = AB^2 + AC^2 + 2BA.AD$.

Note. If **BE** were drawn ⊥ to **CA** meeting **CA** produced in **E**, it could be similarly proved that $BC^2 = AB^2 + AC^2 + 2CA.AE$; so that $BA.AD = CA.AE$. In other words, the product of **BA** and the projection of **AC** upon it = the product of **CA** and the projection of **AB** upon it.

THEOREM 49. [Euclid II, 13.]

181. **In any triangle the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle minus twice the rectangle contained by one of those sides and the projection on it of the other.**

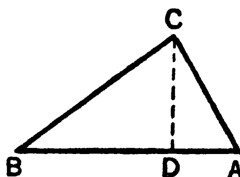


Fig. 420

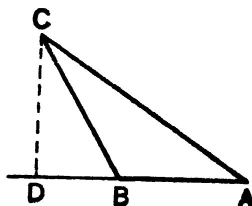


Fig. 421

BAC is an acute angle of the $\triangle$ ABC. CD is the perpendicular from C upon AB (or AB produced); then AD is the projection of AC upon AB.

To prove that $BC^2 = AB^2 + AC^2 - 2AB.AD$.

Proof. Since $\triangle$ BDC is right-angled at D,

$\therefore$ $BC^2 = CD^2 + DB^2$ (*Pythagoras' Theorem.*)

Since $DB = AB - AD$ (in Fig. 420), or $DA - BA$ (in Fig. 421.),

$\therefore$ in both figures, $DB^2 = AB^2 + AD^2 - 2AB.AD$;

$\therefore$ $BC^2 = CD^2 + AB^2 + AD^2 - 2AB.AD$

But $\triangle$ ADC is right-angled at D,

$\therefore$ $CD^2 + AD^2 = AC^2$;

$\therefore$ $BC^2 = AB^2 + AC^2 - 2AB.AD$.

Note 1. *The square on any side* **BC** *of a triangle* **ABC** *is equal to, greater than, or less than the sum of the squares on the other two sides* **(AC, AB)** *according as the angle* **(A)** *opposite to the side* **BC,** *is a*

right angle, an obtuse angle, or an acute angle. When $\angle A$ *is not a right angle the difference between* BC^2 *and* AC^2+AB^2 *is twice the rectangle contained by one of the sides* **AC, AB** *and the projection on it of the other.*

The converse results should also be noted :

(i) If $BC^2=AB^2+AC^2$, $\angle A$ is a right angle. (Theor. 26);

(ii) if $BC^2>AB^2+AC^2$, $\angle A$ is obtuse;

(iii) if $BC^2<AB^2+AC^2$, $\angle A$ is acute.

The formal proofs of the last two results are left to the student.

Note 2. If the sides of a triangle are known we can at once calculate the projection of any side upon any other. Suppose **ABC** is a triangle and **BC** a units of length, $AC=b$ units, $AB=c$ units. Then the projection of **BC** upon $AB=\dfrac{(BC^2+AB^2)-AC^2}{2AB}$

$$=\frac{(a^2+c^2)-b^2}{2c}\text{(units of length)}$$

We have similar formulæ for the projection of **AC** upon **AB**, of **AC** upon **BC** and so on. [See (3) page 296].

Ex. In a $\triangle ABC$, $BC=4$ in., $CA=5$ in., and $AB=7$ in. Is the triangle obtuse angled? Calculate the projections of **AB** and **BC** upon **AC**. Draw a figure and verify by measurement.

[$AB^2-AC^2-BC^2=7^2-5^2-4^2=8$, this shows that $AB^2>AC^2+BC^2$, the angle opposite to **AB** i.e., $\angle C$ is obtuse. The projection of **AB** upon $AC=\dfrac{(AB^2+AC^2)-BC^2}{2AC}$

$$=\frac{74-16}{10}\text{(in.)}$$

$$=5{\cdot}8 \text{ (in.)}$$

The projection of **BC** upon $AC=\dfrac{(AC^2+BC^2)-AB^2}{2AC}$

$$=\frac{41-49}{10}\text{ (in.)}$$

$$=\tfrac{8}{10}={\cdot}8 \text{ (in.)}]$$

EXERCISE LXXVII.

1. Draw any triangle **ABC**, so that the $\angle$**B** is obtuse Draw **AD**, **BE**, **CF** $\perp$ to **BC**, **CA**, **AB** respectively. Verify by measurement the following results,

(i) $AC^2 = AB^2 + BC^2 + 2BC.\ BD.$
$= AB^2 + BC^2 + 2AB.\ BF.$

(ii) $BC^2 = AB^2 + AC^2 - 2AB.\ AF.$
$= AB^2 + AC^2 - 2AC.\ AE.$

(iii) $AB^2 = AC^2 + BC^2 - 2AC.\ CE.$
$= AC^2 + BC^2 - 2BC.\ CD.$

2. The sides of a triangle are 10 cm , 12 cm., and 15 cm. Show without drawing the triangle that it is acute-angled. Find by calculation the projection of each side upon every other. Verify the results by drawing the triangle and taking the necessary measurements.

3. Find whether the triangles whose sides are given below, are acute-angled or not

(i) 7 in., 3 in., 5 in , (ii) 100 ft., 120 ft., 200 ft., (iii) 25 yds., 30 yds., 40 yds

4. In Fig. 420, if **BC**, **CA**, **AB** are respectively *a*, *b*, *c* units of length, calculate the length of **CD** and thence the area of the $\triangle$**ABC**. [See also § 129, II].

5. The sides of a triangle are 4 cm., 5 cm., 6 cm., calculate the lengths of the perpendiculars to the sides from the opposite vertices. Verify by drawing the triangle and measuring the perpendiculars. Calculate also the area of the triangle.

6. In Ex. 5 calculate the radii of the inscribed and escribed circles of the triangle. [See Note 2, § 158 and Ex 4 (LXV).]

7. Calculate the radii of the inscribed circles and escribed circles of the triangles whose sides are, (i) 20, 34, 42, (ii) 40, 45, 50.

8. **ABC** is a triangle, such that **AB**=10 in., **BC**=15 in., and ∠**B**=60°. Calculate the length of **AC**. [See Ex. 12 (LXXVI)].

9. In Ex. 8, if **AB**=12 in., **BC**=15 in, ∠**B**=120°, calculate the length of **AC**.

10. **ABC** is an equilateral triangle each of whose sides is 32 inches long. In **BC** a point **P** is taken 20 inches from **B**, prove that **AP** is 28 inches.

11. **ABC is an isosceles triangle (AB=AC), and P is a point in BC, show that $AP^2 = AB^2 - BP.PC$.**

12. **ABC is an isosceles triangle (AB=AC), and P is a point in BC produced, show that $AP^2 = AB^2 + BP.CP$.[1]**

13. **ABC** is a triangle, with ∠**B**=45°. Prove that the sum of the squares on **AB**, **BC** exceeds the square on **AC** by four times the area of the △**ABC**.

14. **ABCD** is a square. **AC** is produced to **P** such that **CP**=**BC**. Prove that $BP^2 = AC.AP$.

15. **ABCD is any quadrilateral; BP, DQ are perpendiculars upon AC. Prove that the difference between $AB^2 + CD^2$ and $BC^2 + AD^2 = 2AC.PQ$.**

Hence show that if $AB^2 + CB^2 = BC^2 + AD^2$, the diagonals AC.BD are at right angles.

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1. **Exs. 11, 12 constitute what is known as the Theorem of Pappus.**

THEOREM 50.

(Apollonius' Theorem)

182. **In any triangle the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.**

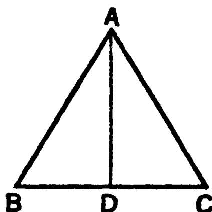


Fig. 422

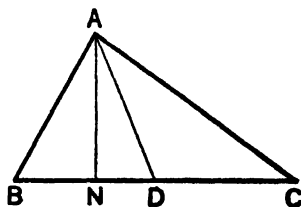


Fig. 423

ABC is a triangle and D is the mid-point of BC.

To prove that $AB^2 + AC^2 = 2BD^2 + 2AD^2$.

Proof. (i) *If* AD *is* ⊥ *to* BC (this will be the case when AB = AC), △s ADB, ADC will be right-angled at D, (Fig. 422),

∴ $AB^2 = BD^2 + AD^2$,

and $AC^2 = DC^2 + AD^2$

But BD = DC

∴ $AB^2 + AC^2 - 2BD^2 + 2AD^2$

(ii) *If* AD *is not* ⊥ *to* BC draw AN ⊥ to BC, (Fig. 423).

Of the two angles ADC, ADB, one is acute, the other is obtuse; suppose ∠ADB is acute.

Then, from △ ADB, since ∠ADB is acute,

∴ $AB^2 = BD^2 + AD^2 - 2BD.DN$. (*Theor.* 49)

From △ ADC, since ∠ADC is obtuse,

∴ $AC^2 = DC^2 + AD^2 + 2CD.DN$. (*Theor.* 48)

But BD = DC.

$\therefore \quad AB^2 + AC^2 = 2BD^2 + 2AD^2$.

Note. The above theorem is usually known as **the Theorem of Apollonius.**

183. In the proof of Theorem 50 if we subtract AB^2 from AC^2, instead of adding them, we get

$$AC^2 - AB^2 = 2CD.DN + 2DB.DN.$$
$$= 2(CD + DB)\ DN$$
$$= 2CB.DN \qquad \text{[See Ex. 22 (LXXV)].}$$

This result may be stated in words as follows :

In any triangle the difference of the squares on any two sides is equal to twice the rectangle contained by the third side and the line between its mid-point, and the foot of the perpendicular drawn to it from the opposite vertex.

184. The following two theorems are deduced at once from the theorems in § 182, 183.

(a) **The locus of a point A such that the sum of the squares of its distances from two fixed points, B, C, is equal to a given square, is a circle having its centre at the mid-point of BC.**

See Fig. 423; $AB^2 + AC^2 = 2BD^2 + 2AD^2$

But $AB^2 + AC^2$ being by hypothesis equal to a given square, is constant; also BD is a fixed length, being half of BC.

$\therefore$ AD is constant,

As D is a fixed point, and AD is constant,

$\therefore$ the locus of A is a circle whose centre is D.

(b) The locus of a point A, such that the difference of the squares of its distances from two fixed points B, C, is equal to a given square, is a straight line perpendicular to BD.

See Fig. 423; $AC^2 - AB^2 = 2CB.DN$ [§ 183].

But $AC^2 - AB^2$ being by hypothesis equal to a given square is constant, and CB is a fixed length,

∴ DN is constant.

But D is a fixed point being the mid-point of BC,

∴ N is a fixed point.

Hence the locus of A is the perpendicular to BC drawn through N. [AC being supposed to be >AB, N is nearer to B than to C and is therefore to the right of D; if however $AC^2 \sim AB^2$ is given to be constant where AC may be greater or less than AB, there is another possible position of the point N on the other side of D; in such a case the locus is a pair of lines ⊥ to BC, one of these lines corresponds to AC>AB, the other to AC<AB].

EXERCISE LXXVIII.

1. Use the Theorem of Apollonius to calculate the length of the medians of a triangle whose sides are 5 cm., 6 cm, 8 cm.; construct the triangle and verify the results by measurement.

2. Use the Theorem of Apollonius to prove that the line joining the vertex to the mid-point of the hypotenuse of a right-angled triangle is equal to half the hypotenuse. [See also Ex. 5, p. 228.]

3. **ABC** is a triangle whose sides **BC, CA, AB** are respectively a, b, c units of length. Find the lengths of the medians **AD, BE, CF**. [$2AD^2 = b^2 + c^2 - \frac{1}{2}a^2$].

4. Two sides of a triangle are 6 in. and 10 in., and the median bisecting the third side is 8 in.; calculate the third-side. Construct the triangle and verify by measurement.

5. If a, b, c are the lengths of the sides of a triangle and l, m, n those of the medians, prove that $4(l^2+m^2+n^2)=3(a^2+b^2+c^2)$.

6. If the medians of a $\triangle$**ABC** meet in **G**, prove that $BC^2+CA^2+AB^2=3(AG^2+BG^2+CG^2)$.

7. Prove that in a right-angled triangle, the sum of the squares on the median is $\frac{3}{2}$ times the square on the hypotenuse.

8. **O** is a point within a rectangle **ABCD**. Prove by the Theorem of Apollonius that $OA^2+OC^2=OB^2+OD^2$. (See also Ex. 22, p. 291).

9. **P** is a point within a rectangle **ABCD** whose diagonals intersect in **O**. Prove that $PA^2+PB^2+PC^2+PD^2=AC^2+4OP^2$.

10. What theorem is deduced from the Theorem of Apollonius by supposing the vertex **A** of the $\triangle$**ABC**, to come nearer and nearer to **BC** until it becomes a point in **BC**, or **BC** produced? [See Ex. 5. (LXXV)].

11. **BC** is a given straight line and **O** any point outside it. If **BC** is trisected at **P** and **Q** prove that $OB^2+OC^2=OP^2+OQ^2+4PQ^2$.

12. **Prove that in any parallelogram the sum of the squares on the diagonals is equal to the sum of the squares on the sides.**

13. **Prove that in any quadrilateral, the sum of the squares on the sides is equal to the sum of the squares on the two diagonals together with four times the square on the line joining the mid-points of the diagonals.**

Hence prove that if the sum of the squares on the sides of a quadrilateral = the sum of the squares on the diagonals, the quadrilateral must be a parallelogram.

14. Find a point **O** in a given straight line **PQ**, such that the sum of the squares of its distances from two given points **A** and **B** may have the least possible value.

15. **ABC is a triangle and D is a point in BC, so that** $p.\,BD=q.\,DC$; **prove that** $p.\,AB^2+q.\,AC^2=p.\,BD^2+q.\,DC^2+(p+q)\,AD^2$.

[Suppose $\angle$**ADB** is acute, ana consequently $\angle$**ADC** is obtuse, then $AB^2=BD^2+AD^2-2BD.DN$ (see Fig. 423), and $AC^2=DC^2+AD^2+2CD.DN$.

[Multiplying the first of these equations by p, the second by q, and adding, and remembering that $p.\ BD=q.\ DC$, we obtain the result at once.].

16. A and B are two points. Find the locus of a point P such that $3\,PA^2+4PB^2$ may be constant [Suppose D is the point in AB such that $3AD=4DB$, then by Ex. 15, $3AP^2+4BP^2 = 3.\ AD^2+4BD^2+7PD^2$.

Since $3\ AP^2+4BP^2$ is to have a constant value, PD must be constant, hence the locus of P is a circle with centre D. Generalise this result.]

17. ABC is a triangle of which the centroid is G; and P is any point. Prove that $PA^2+PB^2+PC^2=GA^2+GB^2+GC^2+3PG^2$.

[Let D, E, F be the mid-points of BC, CA, AB. From $\triangle$ PAB, $PA^2+PB^2=2AF^2+2PF^2$ (by Apollonius' Theorem). Adding PC^2 to both sides, $PA^2+PB^2+PC^2=2AF^2+2PF^2+PC^2$.

Since $2FG=GC$, from $\triangle$ PFC (by Ex. 15), we have

$$2PF^2+PC^2=2FG^2+GC^2+3PG^2$$

$$\therefore\ PA^2+PB^2+PC^2=2AF^2+2FG^2+GC^2+3PG^2.$$

But $2AF^2+2FG^2=GA^2+GB^2$ (from $\triangle$ GAB, by Apollonius' Theorem). Hence $PA^2+PB^2+PC^2=GA^2+GB^2+GC^2+3PG^2$].

18. Find a point such that the sum of the squares of its distances from the vertices of the triangle is the least. [The centroid].

CHAPTER XVII.

RECTANGLES CONTAINED BY SEGMENTS OF CHORDS.

THEOREM 51A. [Euclid III, 35]

185. **If two chords of a circle intersect at a point inside the circle, then the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.**

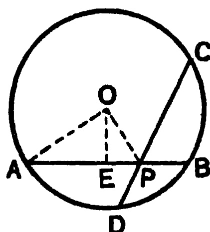


Fig. 424

AB, CD are two chords of a circle intersecting at the point **P** inside the circle.

To prove that **AP. PB = CP. PD.**

Let **O** be the centre of the circle and r its radius; draw **OE** ⊥ to **AB**. Then **E** is the mid-point of **AB**. Join **OA, OP**.

Proof. $AP.PB = (AE + EP)(EB - EP)$

$= (AE + EP)(AE - EP)$

$= AE^2 - EP^2$ (*Theor.* 47)

$= (AE^2 + OE^2) - (EP^2 + OE^2)$

$= OA^2 - OP^2$ (∵ △s **OEA, OEP** are right-angled at **E**)

$= r^2 - OP^2$.

Similarly, CP.PD $= r^2 - \text{OP}^2$.

$\therefore$ AP.PB = CP.PD.

Corollary 1. *If P be a point inside a circle with centre* O *and radius r, then the rectangle contained by the segments of any chord through* P *(regarded as divided at* P*) is equal to* $r^2 - \text{OP}^2$.

Corollary 2. *The rectangle contained by the segments of any chord through* P *is equal to the square on half the chord which has* P *for its mid-point*

Ex. 1. P is a point 2 cm. from the centre of a circle of diamerer 10 cm. Find the area of the rectangle contained by the segments of any chord through P.

Ex. 2. P and Q are two points within a circle equidistant from the centre. Proved that the rectangle contained by the segments of any chord through P is equal to that contained by the segments of any chord through Q.

186. A problem. To find the fourth proportional to three given lengths.

Let a, b, c, be the three given lengths. To find a length x such that $a : b :: c : x$. i.e., the rect. ax = rect. bc.

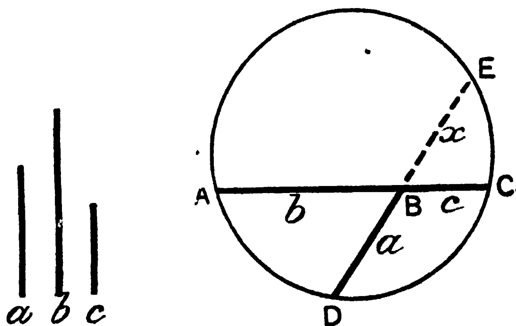


Fig. 425

Draw any straight line ABC taking AB $= b$, BC $= c$. Through B draw any other straight line BD $= a$. Draw a

circle through **A, C, D.** Produce **DB** to meet the ⊙ in **E.** Then **BE** is the required fourth proportional to *a*, *b*, *c*; for **DB. BE = AB. BC.** [Theor. 51A]

Note. If $a : b = b : x$, then x is said to be a **third proportional** to a and b. If in the above construction (Fig. 425) we take $c = b$, we obtain the third proportional to a and b.

Ex 1. Construct the fourth proportionals to the following sets of three lengths. (i) 1·2 in., 1·7 in., 2 in. (ii) 1·5 in., 0·8 in., 2·2 in. (iii) 4 cm., 3 cm., 6 cm.

Ex. 2 Find graphically the fourth proportionals to the following numbers, (i) 2, 3, 5, (ii) 5, 3, 6, (iii) $2\frac{1}{2}$, $1\frac{1}{4}$, 3.

Ex. 3. Find the third proportionals to (i) 2 in., 3 in.; (ii) 4 cm., 3·5 cm.

THEOREM 51B. (Euclid III, 36).

187. If two chords of a circle intersect at a point outside the circle, then the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

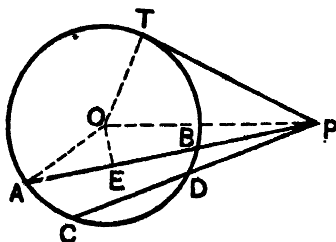


Fig. 426

AB, CD are two chords of a circle intersecting at the point **P** outside the circle.

To prove that **PA. PB = PC. PD**

Let **O** be the centre of the circle and *r* its

radius. Draw **OE** $\perp$ to **AB**, then **E** is the mid-point of **AB**.

Join **OA, OP.**

Proof. $PA.PB = (PE + EA)(PE - BE)$

$= (PE + EA)(PE - EA)$

$= PE^2 - EA^2$ (*Theor.* 47)

$= (PE^2 + OE^2) - (EA^2 + OE^2)$

$= OP^2 - OA^2$. [$\because$ $\triangle$s OEP, OEA are right-angled at E]

$= OP^2 - r^2$.

Similarly $PC.PD = OP^2 - r^2$.

$\therefore$ $PA.PB = PC.PD$.

Corollary 1. *If* **P** *be a point outside a circle with centre* **O** *and radius r, then the rectangle contained by the segments of any chord passing through* **P** *(regarded as divided at the point* **P***) is equal to* $OP^2 - r^2$.

Corollary 2. **If PT** (see Fig. 426) **be a tangent to a circle drawn from a point P outside the circle, and AB any chord passing through P then** $PT^2 = PA.PB$. [Join **OT** (Fig. 426). Since $\angle$**OTP** is a rt. $\angle$, $PT^2 = OP^2 - OT^2 = OP^2 - r^2$].

The result may also be deduced from Theorem 51B, by supposing the secant **PBA** to turn round **P** (Fig. 427), until it comes to the position **PT** and becomes a tangent to the $\odot$. In such a position each of the segments **PA, PB** becomes **PT**, and we have $PA.PB = PC.PD = PT.PT = PT^2$.

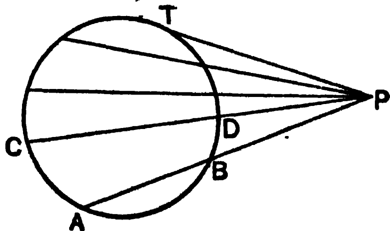


Fig. 427

188. We shall now consider the converse of Theorems 51A, 51B.

(a) If two straight lines intersect, or both being produced intersect, so that the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other, the extremities of the lines are concylic.

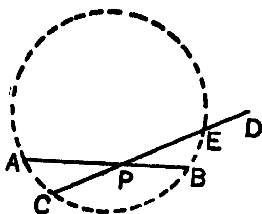


Fig. 428

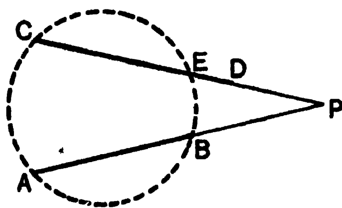


Fig. 429

The straight lines **AB**, **CD** intersect at **P** (Fig. 428) so that **AP**. **PB** = **CP**. **PD**;

or they intersect, when both are produced, at **P** (Fig. 429) such that

PA.PB = PC.PD

To prove that in either case the four points **A**, **B**, **C**, **D** are concyclic.

Proof. A circle can be drawn through the three points **A**, **B**, **C**. This circle will pass through **D**.

If not, let the circle cut **CD** (or **CD** produced) again in some point E other than **D**.

Then because the chords **AB**, **CE** intersect at **P**,

∴ **AP.PB = CP.PE** [*Theorem* 51A or 51B],

But **AP.PB = CP.PD** (*given*);

∴ **CP.PE = CP.PD**

∴ **PE = PD**,

which is impossible, because of PE, PD, one is a part of the other.

∴ the ⊙ drawn through A, B, C will pass through D.

(b) If a chord AB of a circle is produced to any point P, and from P a straight line PT is drawn to meet the circle in a point T, so that the square on PT is equal to the rectangle contained by the segments PA, PB, then PT is a tangent to the circle.

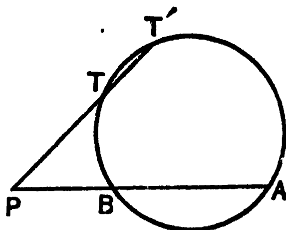


Fig. 430

Proof. If PT be not a tangent to the ⊙ at T, let it meet the circle again in T′.

Since the chords T′T, AB intersect at P,

PT.PT′ = PB.PA. [*Theor.* 51 B.]

But PT^2 = PB.PA, (*given*)

∴ PT.PT′ = PT^2;

∴ PT′ = PT.;

which is impossible, because of PT, PT′ one is a part of the other.

∴ PT is a tangent to the circle.

EXERCISE LXXIX.

1. Calculate the areas of rectangles contained by the segments of the chords of a circle (centre O), passing through a point X, when (i) r (i.e., radius) = 2 in., OX = 1·5 in.; (ii) r = 6 in, OX = 4 in.; (iii) r = 10 cm., OX = 15 cm.; (iv) r = 10 cm. OX = 12 cm., (v) r = 10 cm., OX = 8 cm.

2. The radius of a circle is 4 cm., and its centre is **O**. **P** is a point inside the circle at a distance of 2 cm. from **O**. **Q** is a point outside the circle, such that the rectangle contained by the segments of any chord through **P** is equal to the rectangle contained by the segments of any chord through **Q**. Calculate the distance of **Q** from **O** to the nearest millimetre.

3. Draw two lines **AXB**, **CXD**, intersecting at **X**; make **XA** = 1·5 in., **XB** = 2 in., and **XC** = 1 in. Draw the circle passing through **A**,**B**,**C**, and let it cut **XC** again at **D**. Calculate **XD**, and check by measurement.

4. Two straight lines **AB**, **CD** intersect at **Y**, when both of them are produced. If **YA** = 8 cm., **YB** = 6 cm., **YC** = 12 cm., and if the circle through **A**, **B**, **C** cut **YC** again at **D**, calculate **YD** and verify by measurement.

5. **ABC is a triangle right-angled at A. AD is drawn ⊥ to BC. Prove that AD^2 = BD.DC.** [Let **AD** produced cut the circumcircle of △ **ABC** in **E**, then **AD.DE** = **BD.DC**. See also Ex. 11, (LXXV)].

6. **The tangents to two intersecting circles from any point on their common chord *produced* are equal.**

7. Prove that the common tangents to two intersecting circles are each bisected by the common chord (produced).

8. **O**, **A**, **B**, are three points *in order* on a straight line. Prove that the tangents from **O** to all circles which pass through **A** and **B** are equal.

9. **O**,**A**,**B** are three fixed points *in order* on a straight line. A tangent **OT** is drawn to a variable circle which passes through **A** and **B**, touching the circle at **T**. Find the locus of **T**.

10. If the tangents from a point **O** to two intersecting circles are equal, prove that **O** must be on the common chord produced. (Let **A** be a point in which the circles intersect. Join **OA** and produce it to cut the circles again in **X**, **Y**. Then **OA.OX** = **OA.OY**. ∴ **OX** = **OY**. Hence **X**, **Y** must be the same point and this point will be common to both the ⊙s.)

11. **Find the locus of a point P such that tangents from it to two given intersecting circles may be equal.**

12. **ABC** is a triangle right-angled at **A**; **AD** is drawn perpendicular to **BC**. Prove that $AC^2 = CB.CD$. [**CA** is a tangent to the circumcircle of the △**ABD**].

13. **ABC** is any triangle. **BE, CF** are drawn ⊥ to **CA,AB** respectively, intersecting at **P** Prove that (i) **BP. PE=CP. PF** (ii) **AB. AF=AC. AE** (iii) **BP. BE=BF. BA.**

14. Through any point **O** in the common chord of two circles, a straight line is drawn cutting one circle in **P, Q** and the other in **R, S.** Prove that **PO. OQ=RO. OS.**

15. **AB** is a diameter of a circle and **C** is a given point in **AB**, or **AB** produced Through **C** a perpendicular to **AB** is drawn cutting any chord **AP** through **A** in **Q.** Prove that **AP. AQ** is constant. (**AP. AQ=AC AB**).

16. **CD** is a fixed straight line and **A** is a fixed point outside **CD. A** is joined to any point **Q** in **CD**, and on **AQ** or **AQ** produced, a point **P** is taken such that **AP. AQ** is constant. Prove that the locus of **P** is a circle passing through **A.**

17. **From A, tangents AP, AP′ are drawn to a circle whose centre is O and radius is r; OA cuts the chord of contact PP′ in A′. Prove that OA.OA′ = r^2.**

Note. The points **A** and **A′** are said to be a pair of **inverse points** with respect to the circle. The two points are such that **OA.OA′**=(radius)^2 and **AA′** passes through the centre of the circle.

18. **A, B, C, D** are four points on a given straight line. Find a point **O** in the line such that **OA. OB=OC.OD.**

[Take any point **P** outside the given line Draw the ⊙s **APB** and **CPD** cutting again in **Q**. Let **PQ** cut the given line in **O.** Then **O** is the required point].

19. **Every circle which passes through a pair of inverse[1] points A, A′, with respect to a given circle (centre O) cuts the given circle orthogonally, (i.e., at right angles).** [See § 154].

[Let a circle through **A, A′** cut the given ⊙ in **P.** Join

OP. Then since **A, A′** are inverse points with respect to the given circle, $OP^2 = OA \cdot OA'$. (see note, Ex. 17)

OP is a tangent to the circle through **A, A′**. Also the tangent **PC** to the given ⊙ at **P** is ⊥ to **OP**. Thus the tangents to the two circles at the point of intersestion (**P**) are at right angles; hence the two circles cut orthogonally]

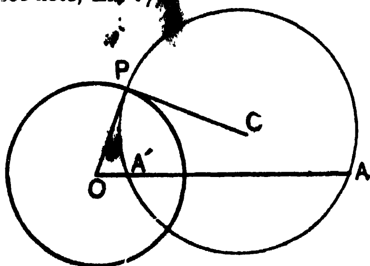


Fig. 431

20. **Construct a circle to pass through two given points A and B, and touch a given circle (centre O)**

Through **A, B** draw any circle (shown by a dotted line in the figure), of a suitable radius, so as to cut the given circle in two points, say **P** and **Q** Join **AB** and **PQ** and let them meet at **R** From **R** draw **RS** touching the given circle at **T**. The circle drawn through the points **A, B, T** is the required circle. **Proof** —**RT** is a tangent and **PQ** a chord of the given circle, ∴ $RT^2 = RP.\ RQ$. Again **PQ, AB** are chords of the dotted circle meeting in **R**, $RP.RQ = RA.RB$;

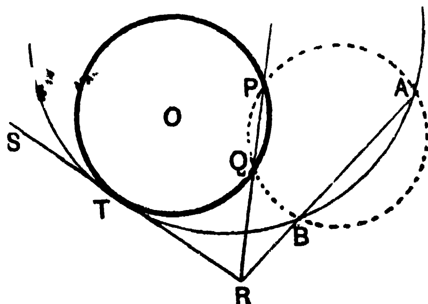


Fig. 432

$RT^2 = RA.RB$. Hence **RT** touches the circle drawn through **T, A, B** at **T** [§ 188 (b) | **RT** also touches the given circle at **T**. Hence the ⊙ through **T, A, B,** touches the given circle at **T**.*

* As another tangent can be drawn from **R** to the given ⊙, there is another circle satisfying the given conditions. If the perpendicular bisector of **AB** passes through **O** (the centre of the given ⊙, then **AB** and **PQ** will be parallel, so that **R** will be a point at infinity, and the tangent **RT** will be parallel to **AB**. The student is advised to solve the problem completely in such a case.

CHAPTER XVIII.

MISCELLANEOUS APPLICATIONS.

189. **To construct a square equal in area to a given rectangle.**

It is required to construct a square equal in area to the given rectangle ABCD.

Produce AB to E making BE = BC. Bisect AE at O, and with centre O and radius OA describe a circle[1] cutting CB produced at P.

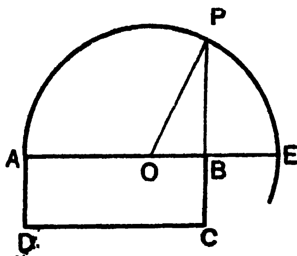


Fig. 433

Then the square on BP = rect. ABCD.

Join OP.

Proof. Rect. ABCD = AB. BE.

= (AO + OB) OE − OB)

= (AO + OB) (AO − OB)

= $AO^2 - OB^2$.

= $OP^2 - OB^2$.

= PB^2. (∵ ∠OBP = a rt. angle).

Note that $AB.BE = OA^2 - OB^2$ ollows at once form Cor. to Theor. 46.

Corollary. If from any point P on a circle a perpendicular PB is drawn to any diameter AE, (see Fig. 433) **meeting the diameter in B, then AB.BE** = BP^2. See also Ex. 11 (LXXV) and Ex. 5 (LXXIX)]

[1] This circle will pass through E.

190. Applications.

(i) To find the mean proportional between two given straight lines. (This is the same problem as to find a line the square on which is equal to the rectangle contained by the two given lines). Let the given lines AB, BE be placed in one straight line, (see Fig. 433). On AE as diameter describe a ⊙. Draw BP ⊥ to AE to meet the ⊙ in P. Then PB is the mean proportional between AB, BE. [For $PB^2 = AB.BE$.

Note. If a, x, b, are three quantities such that $a : x = x : b$, then x is called the **mean proportional** between a and b. It is easily seen that $x^2 = ab$ or $x = \sqrt{ab}$. Thus the mean proportional between two given quantities is the square root of their product. The above problem indicates a graphical method for finding the mean proportional between two given numbers. [See Ex. 4 LXXX].

(ii) To describe a square equal in area to a given rectilinear figure. Make a triangle equal in area to the given figure (see § 121), next make a rectangle equal in area to the triangle (see § 122), lastly make a square equal in area to the rectangle (§ 189).

EXERCISE LXXX.

1. Construct squares equal in area to the figures mentioned below, and in each case measure a side of the square:

(i) a rectangle of sides 4cm. and 5 cm ;

(ii) a rectangle whose area is 30 sq. cm. (consider a rectangle of sides 5 cm. and 6 cm.) ;

(iii) an equilateral triangle of side 2 in. ;

(iv) a triangle ABC, in which $\angle B = 40°$, $\angle C = 70°$, and BC = 2 in. ;

(v) a triangle whose sides are 4 cm., 6 cm. and 7 cm. ;

(vi) a parallelogram **ABCD** in which **AB** = 1·5 in., **AD** = 1 in, and ∠**A** = 45° ;

(vii) a regular pentagon of side 1·2 in. ;

(viii) a regular hexagon of side 3 cm. ;

(ix) the quadrilateral **ABCD** in Ex 3, p 87.

2. To construct a square equal to a given parallelogram.

3. Construct the mean proportionals between the lengths (i) 2 in, 3 in., (ii) 1·5 in., 2 in., (iii) 4·2 cm, 6·1 cm.

4. Find graphically the mean proportional between 3 and 4 [Find the mean proportional between 3 units of length and 4 units of length, say 3 cm. and 4 cm. The measure of this mean proportional in centimetres is the mean proportional between the numbers 3 and 4]

5. Find graphically $\sqrt{21}$. [Find the mean proportional between 7 and 3].

6. Find graphically $\sqrt{11}$, $\sqrt{15}$, $\sqrt{27}$, $\sqrt{56}$.

7. Construct a square of area 24 sq. cm. without calculating beforehand the length of a side of the square

8. Find a length such that the rectangle contained by this length and a given length may be equal to the square on another given length

9. **To construct a circle which shall pass through two given points A, B, and touch a given straight line PQ.**

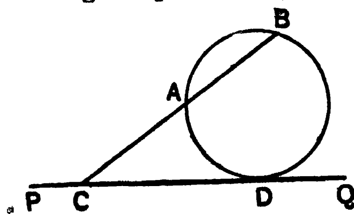


Fig. 434

[Join **BA** and produce it to meet **PQ** in **C**. Find the point **D**, such that $CD^2 = CA.CB$ (§ 189). Draw a circle through the points **A, B, D**. This circle passes through **A, B**, and touches **PQ** at **D** [§ 188 (b)].

191. (a) To divide a given straight line AB internally at P such that the rect. AP.PB may be equal to a given square (of side k).

With **AB** as diameter describe a circle. Draw **BN** ⊥ to **AB**, and equal to k. Through **N** draw a parallel to **BA** meeting the ⊙ in **H**.

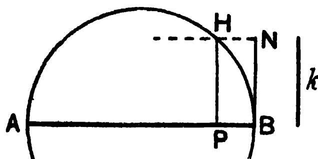


Fig. 435

Draw **HP** ⊥ to **AB**.

Then $AP.PB = HP^2 = k^2$ (*Corollary*, § 189)

Note. The problem fails if the parallel to **BA**, drawn through **N** does not cut the ⊙ described on **AB** as diameter; this will be the case when $k > \frac{1}{2}$ **AB**. The greatest possible value of k is $\frac{1}{2}$ **AB**; in this case **P** is at the mid-point of **AB**.

Corollary. The rectangle AP.PB is greatest when P is the mid-point of AB (See Ex. 1, LXXV).

(b) To divide a given straight line AB externally at P such that the rect. AP·BP may be equal to a given square (of side k).

[If **AB** is produced to any point **P**, and **C** is the mid-point of **AB**, then $AP.BP = CP^2 - CB^2$ (see Cor., § 176)]. This analysis gives the clue to the construction.

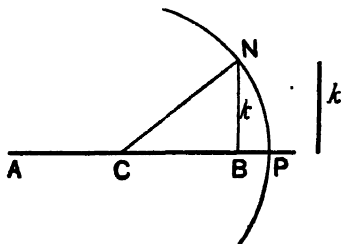


Fig. 436

Bisect **AB** at **C**. Draw **BN** ⊥ to **AB** and equal to k. Join **CN**. With centre **C** and radius **CN** describe a ⊙ cutting **AB** produced in **P**. Then

$AP.BP = CP^2 - CB^2 = CN^2 - CB^2 = BN^2 = k^2$ (Theor. 25).

192. Applications to Algebra.

Ex. 1. Find two numbers x and y such that $x+y=4$, and $xy=2$.

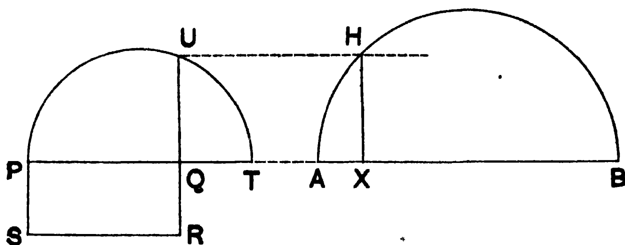


Fig. 437 Fig. 438

Draw a rectangle **PQRS**, with **PQ**=2cm. and **QR**=1cm so that the area of the rectangle=2 sq. cm. By § 189 obtain the line **QU**, (Fig. 437) the square on which=rect **PQRS**, and hence is of area, 2 sq. cm. Take a line **AB**=4 cm. (better in a straight line with **PQT**).[1] By § 191 (*a*) obtain the point **X** in **AB**, (Fig. 438) such that **AX.XB**=the square on **QU**. Hence **AX.XB**=2 sq. cm. Also **AX**+**XB**=4 cm Thus the measures of **AX**,**XB** in centimetres are the required values of x, y These measures are found (by actual measurement) to be 0·6 and 3·4 approximately.

Note. If it were given that $x+y=4$, $xy=5$ no two numbers[2] x, y could be found so as to satisfy the above conditions; for the given value of xy (i. e. 5) is greater than the square of half the given sum of x and y i.e. $(\frac{4}{2})^2$ [see note § 191 (*a*)].

Ex. 2. Find two positive numbers x. y. such that $x-y=4$, and $xy=2$

---

1. The reason for taking **AB** in a straight line with **PQT** is that the point **H** is then obtained by simply drawing a parallel to **PT** through **U**.

2. Excluding imaginary numbers from consideration.

As in example 1, you would find **QU** (Fig. 437) such that the square on **QU** = 2 sq. cm., Then by § 191 (*b*) you would find a point **X** in **AB** produced such that **AX**, **XB** = $\mathbf{QU}^2$ = 2 sq. cm. (Fig. 439). Also **AX** − **BX** = **AB** = 4cm. The measures of **AX**, **XB** in centimetres are the required values of x, y. These measures are approximately 4·45 and 0·45.

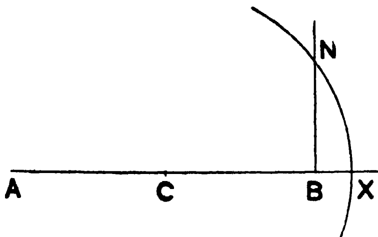


Fig. 439

193. Graphical solution of quadratic equations. It is a well-known property that if a, b are the roots of the quadratic equation,

$$x^2 + px + q = 0, \qquad \text{(i)}$$

then

$$\left.\begin{array}{l} a + b = -p \\ ab = q \end{array}\right\} \qquad \text{(ii)}$$

As every quadratic equation can be reduced to the form (i), the relations (ii) indicate that the equation may be solved graphically as explained in § 192. If in any particular case the construction fails, the inference will be that the equation has no *real* roots.

Ex. 1. Solve $x^2 - 4x + 2 = 0$. If a, b are the roots then $a + b = 4$, and $ab = 2$. If the roots are real, then since $ab = 2$ (positive), a and b must be both positive or both negative. But since $a + b$ is positive (being equal to 4), a and b must be both positive. Thus a and b are two positive numbers which satisfy the conditions, $a + b = 4$, $ab = 2$, and may be found as in Ex. 1, § 192.

Ex. 2. Solve $x^2 - 4x + 5 = 0$. The construction fails in this case (see note Ex. 1, § 192); hence the roots cannot be real.

Ex. 3. Solve the equation $x^2 + 4x - 2 = 0$. Here $a + b = -4$, and $ab = -2$. These relations indicate that a and b must be of opposite

signs (since ab is negative), and the difference and the product of their *numerical* values are respectively 4 and 2. Suppose p and q care the numerical values of the roots, and p is greater than q, then p—q=4 and pq=2 ; p and q may be found as in Ex. 2, § 192. The roots of the given equation are $-$p, and q [as $a+b$ is negative, the negative root must be of greater numerical value.]

Ex. 4. Solve the equation $x^2+4x+2=0$

Here $a+b=-4$, $ab=2$ These relations indicate that a and b are both negative. The sum and product of the *numerical* values of the roots are respectively 4 and 2. Hence the numerical values of the roots may be obtained as in Ex. 1, § 192. If they are p and q, the roots are $-p$, $-q$

EXERCISE LXXXI.

1. Draw a straight line 2·5 in. long. Divide it *internally*, such that the rectangle contained by the segments may be equal to a square whose side is 1 in.

2. Can you divide a straight line 8 in. long *internally* into two segments such that the rectangle contained by the segments may be equal to a square (i) whose side is 6 in. (ii) whose area is 20 sq. in. ? Give reasons for your answers.

3. Draw a straight line 3 in. long, and divide it *externally*, so that the rectangle contained by the segments may be equal to a square (i) whose side is 1 5 in. (ii) whose area is 6 sq. in.

4. Solve graphically the following equations,
   (i) $x+y=5$, $xy=4$;
   (ii) $x+y=6$, $xy=7$;
   (iii) $x-y=3$, $xy=5$;
   (iv) $y-x=4$, $xy=10$.

5. Solve graphically the following quadratic equations,
   (i) $x^2-5x+6=0$, (ii) $x^2+7x+12=0$,
   (iii) $2x^2-5x+2=0$, (iv) $2x^2+3x-2=0$.

194. (a) To find a point P in a given straight line AB, or in AB produced, such that $AP^2 + PB^2$ may be equal to a given square (of side k).

At B draw BL making $\angle ABL = 45°$. With centre A and radius $= k$ draw a circle cutting BL in H. Draw HP ⊥ to AB. Then $AP^2 + PB^2 = k^2$. [The proof is left to the student]. It should be noted that

Fig. 440

(i) if $k < AN$ (i.e., the perp. from A on BL), the ⊙ with centre A and radius $= k$, will not cut AB, and the problem fails;

(ii) if $k > AN$ but $< AB$, the ⊙ cuts BL in two points H, H′ (Fig, 440), and the perpendiculars from these points to AB meet AB in points P, P′, both lying *between* A and B; and we shall have $AP^2 + PB^2 = k^2$ and $AP'^2 + P'B^2 = k^2$; thus apparently there are two solutions, but it may be proved that $AP = BP'$ and $BP = AP'$.

(iii) If $k > AB$, the ⊙ cuts BL in two points H, H′ equidistant from N, one of the points of intersection H being on LB produced (Fig. 441). The perpendiculars from H, H′ to AB meet AB produced, and BA produced at P, P′ respectively, and it may be shown that $AP^2 + PB^2 = k^2$, also $AP'^2 + BP'^2 = k^2$; thus apparently there are two solutions, but it may be

Fig. 441

shown that $AP = BP'$ and $BP = AP'$

Interpret the case when $k = AN$. (Note that $AB^2 = 2AN^2$).

(b) **Find a point P in a given straight line AB, or in AB produced, such that $AP^2 - PB^2$ may be equal to a given square (of side k).**

At B draw BC ⊥ to AB, making $BC = k$. From C draw CP meeting AB in P, such that $\angle ACP = \angle BAC$; then P is the required point, so that $AP^2 - BP^2 = k^2$.

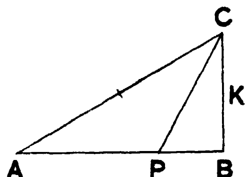


Fig 441 (a)

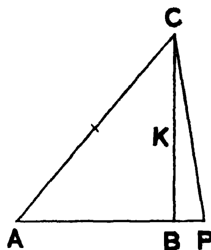


Fig. 441 (b)

The proof is left to the student. Note the two cases, viz., (i) when $k < AB$, P is between A and B. [Fig. 441 (a)], (ii) when $k > AB$, P is on AB produced [Fig. 441 (b)]. Consider the case when $k = AB$.

EXERCISE LXXXII

1. **AB** is a straight line 6 cm long. Find a point **P** in it such that $AP^2 + PB^2$ may be equal in area to a rectangle of sides 4 cm and 7 cm.

2. Solve graphically.
   (i) $x + y = 5$ and $x^2 + y^2 = 16$.
   (ii) $x + y = 6$ and $x^2 + y^2 = 24$.

3. Solve graphically,
   (i) $x - y = 4$ and $x^2 + y^2 = 20$.
   (ii) $x - y = 3$ and $x^2 + y^2 = 12$.

4. **AB** is a straight line 10 inches long, can you divide it internally at a point **P** such that AP^2+PB^2 may be equal to a square (i) of side 6 in., (ii) of side 8 in.

5. Solve graphically

(i) $x+y=6.\quad x^2-y^2=16$

(ii) $x-y=4.\quad x^2-y^2=25.$

Verify the results by solving algebraically.

195. Medial Section. To divide a given straight line AB (internally or externally) into two segments such that the rectangle contained by the whole line and one segment may be equal to the square on the other segment. [Euclid II. 11 also VI. 30.]

(i) To divide AB *internally* at P such that AP.PB $=AP^2$. On AB describe the square ABCD. Bisect AD at E and join BE. Produce EA to F making EF = EB. On AF describe the square AFGP as shown in Fig. 442. (Note that AP, AB are in one st. line). Then AB. PB $=AP^2$. Produce GP to meet DC at H.

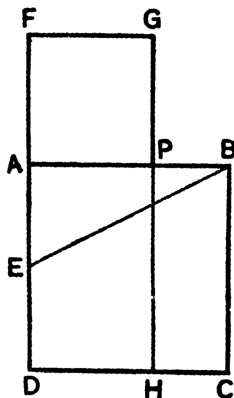


Fig. 442

Proof. Since DA is bisected at E and produced to F,

$\therefore$ DF. AF $=EF^2-EA^2$ (See Cor., § 176)

$=EB^2-EA^2$

$=AB^2$ (*Pythagoras'* (*Theorem*);

that is, rect. FH = sq. AC;

Taking away the common part, the rect. AH, we have sq. FP = rect. PC,

i.e., AP^2 = AB.PB.

(ii) To divide AB *externally* at **P** such that $AP^2 =$ AB.PB. Proceed as in (i), only instead of producing **EA**, produce **ED** to **F** making **EF = EB**: also the square on **AF** is to be described on the side of **AF** *opposite* to the square **ABCD** as shown in Fig. 443.

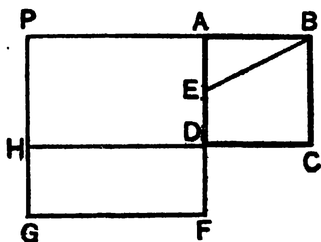


Fig. 443

It may be shown that $AP^2 = AB.BP$.

The proof is left to the student.

A straight line is said to be divided in **medial section** (or in *aurea sectione*, or in **extreme and mean ratio**) when the rectangle contained by the whole line and one of the segments is equal to the square on the other segment.

196. Alternative method for dividing a given line AB in medial section.

Draw **BC** ⊥ to **AB** and equal to $\frac{1}{2}$ **AB**. Join **AC**. With centre **C** and radius **CB** describe a ⊙ cutting

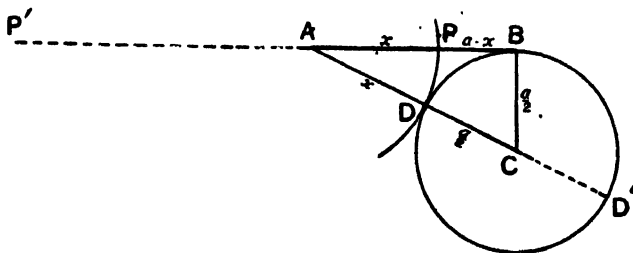


Fig. 444

AC at **D**. With centre **A** and radius **AD**, draw a ⊙ cutting **AB** at **P** (Fig. 444).

Then AB is divided *internally* in medial section at P, i.e. $AP^2 = AB.PB$.

Proof. Let $AB = a$ and $AP = x$,

then $BC = \frac{a}{2}$; $AD = x$; $PB = a - x$, and $AC = x + \frac{a}{2}$

Now $AC^2 = AB^2 + BC^2$; (*Pythagoras' Theorem*).

i.e. $\left(x + \frac{a}{2}\right)^2 = a^2 + \left(\frac{a}{2}\right)^2$;

or $x^2 + ax + \frac{a^2}{4} = a^2 + \frac{a^2}{4}$;

$\therefore$ $x^2 = a^2 - ax = a(a - x)$,

i.e. $AP^2 = AB.PB$.

The circle with centre C and radius CB cuts AC *produced* in D′ (see Fig. 444). If we draw a ⊙ with centre A and radius AD′, this ⊙ will cut BA produced in P′. It may be shown that $AP'^2 = AB.P'B$, that is to say, AB is divided *externally* in medial section at P′. The proof is left to the student.

Note. Algebraic method. If x be the length of the segment **AP** (Fig. 444) and a the length of **AB**, then x satisfies the equation,

$$x^2 = a(a - x),$$

$$\text{i.e. } x^2 + ax - a^2 = 0.$$

Hence the distance **AP** may be obtained by solving this equation. The equation has *two* roots, being a quadratic equation; and since the product of the roots $(= -a^2)$ is negative, the roots must be of opposite signs. The positive root gives the point of internal medial section, and the negative root gives the point of external medial section.

EXERCISE LXXXIII.

1. Draw a straight line **AB** 2 inches long, and divide it in medial section *internally* at **P** and *externally* at **P′**, such that $AP^2 = AB.PB$, and $AP'^2 = AB.P'B$.

2. In Ex. 1 divide **BA** in medial section *internally* at **Q** and *externally* at **Q′** such that $BQ^2 = AB.AQ$ and $BQ'^2 = AB.AQ'$.

3. In Exs. 1, 2 show by measurement that $AP = BQ$, and $AP' = BQ'$.

4. In Ex. 1, measure with a diagonal scale the distances **AP**, **AP′** correct to the nearest hundredth of an inch, and verify the result algebraically by solving a certain quadratic equation.

5. If any straight line **AB** be divided internally and externally in medial sections at **P** and **P′** respectively, such that $AP^2 = AB.PB$, and $AP'^2 = AB.BP'$, prove algebraically that $\frac{AP}{AB} = \frac{\sqrt{5}-1}{2}$ and $\frac{AP'}{AB} = \frac{\sqrt{5}+1}{2}$. (Solve the equation $x^2 + ax - a^2 = 0$, where a is the length of **AB**).

Use algebraic methods to solve the following problems.

6. Divide a given straight line **AB** into two parts such that the square on one part may be equal to 5 times the square on the other part. Consider the particular case when $AB = 2$ in.

7. Divide a given straight line **AB** into two parts such that the rectangle contained by the whole line and one part may be equal to four times the square on the other part

8. If a straight line **AB** is divided internally in medial section at **P** such that $AP^2 = AB.PB$, prove that $PB = \frac{\sqrt{5}-1}{2}AP$.

9. In Ex. 8, show that if from the segment **AP** a part **AQ** be cut off equal to the smaller segment **PB**, then **AP** is divided in medial section at **Q**.

10. If **AB** is divided in medial section at **P**, show that the rectangle contained by the segments **AP**, **PB** is equal to the rectangle contained by the sum and difference of the segments.

197. To construct an isosceles triangle so that each of the angles at the base is double the vertical angle.

Draw any straight line AB and divide it in medial section at P (§ 196), such that $AP^2 = AB.PB$. (This construction is shown separately in Fig. 445.)

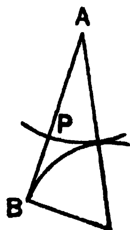


Fig. 445

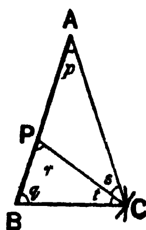


Fig. 446

With centres B and P and radius = AP draw two ⊙ˢ meeting at C (Fig. 446). Join AC, BC. Then ABC is the required triangle. Join PC.

Proof. Since PA = PC, ∴ $\angle s = \angle p$.

Since PC = BC, ∴ $\angle q = \angle r$.

But $\angle r = \angle p + \angle s = 2\angle p$. (*Theor.* 16 *Cor.*)

∴ $\angle q = 2\angle p$.

Again since $BP.BA = AP^2 = BC^2$,

∴ BC is a tangent to the ⊙ passing through A, P, C. (§ 188 b).

∴ $\angle t = \angle p$. [*Theor.* 43].

∴ $\angle ACB = \angle s + \angle t = 2\angle p$.

Thus the ∠ˢ ABC, ACB are each double ∠BAC.

Note. Since ∠ˢ A, B, C are together = 180°, and ∠B, ∠C are each = 2∠A.

∴ $5\angle A = 180°$

∴ $\angle A = 36°$

and $\angle B = \angle C = 72°$.

Thus, with the aid of ruler and compasses only, we are in a position to construct an angle of 36° or 72°. As any angle may be bisected with the aid of ruler and compasses, we can construct angles of 18°, 9°, $4\frac{1}{2}$° etc. by a process of successive bisections. This shows that regular polygons of 5 sides, 10 sides, 20 sides and so on can be inscribed in a circle with the aid of ruler and compasses only. (See Note, § 164).

It may be observed that in Fig. 446, since ∠BAC=36°, BC is a side of a regular decagon inscribed in the circle drawn with centre A and radius AB.

Ex. Construct angles, 72°, 36°, 18°, 9°, without the aid of a protractor.

198. To inscribe a regular pentagon in a given circle. (Euclid. IV, 11.)

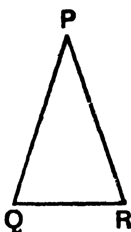


Fig. 447

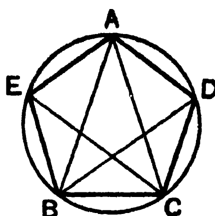


Fig. 448

Construct an isosceles triangle PQR (Fig. 447), whose base angles Q, R are each double the vertical angle P. (See § 197.)

In the given ⊙ (Fig. 448) inscribe △ABC equiangular with △ PQR, so that ∠BAC=∠P, ∠s ABC, ACB equal, to ∠s Q, R (see § 161).

Thus each of the ∠s ABC, ACB is double ∠BAC.

Draw BD, CE bisecting ∠s ABC, ACB respectively, meeting the circle again in D and E.

Join **AE, EB, AD, DC.** Then **AEBCD** is the required regular pentagon.

Proof. Since ∠ˢ **ABC, ACB** are double ∠**BAC**,
∴ ∠ˢ **ACE, ECB, BAC, CBD, DBA** are equal ;
∴ arcs **AE, EB, BC, CD, DA** are equal (Cor. 1, § 148).
Hence **AEBCD** is a regular pentagon. (See § 164, I).

199. **To inscribe in a given circle a regular quindecagon** (**i.e., a regular polygon of fifteen sides**). (Euclid IV, 16).

Let **AC** be a side of an equilateral triangle, and **AB,** a side of a regular pentagon inscribed in the given circle.

Of the fifteen equal arcs into which the whole circumference is divided by the vertices of a regular quindecagon inscribed in the circle, there are three in the arc **AB,** and five in the arc **AC,** hence two in the arc **BC.**

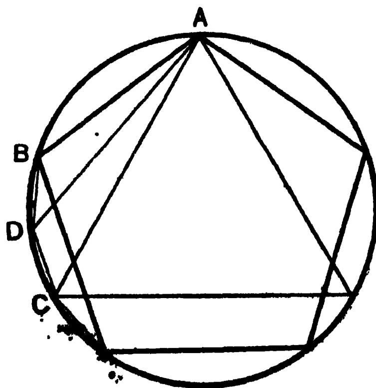


Fig. 449

If we bisect ∠**BAC** by the line **AD** meeting the ⊙ in **D,** then **D** is the mid-point of arc **BC** (see Cor. 1, *Theor.* 38).

Hence each of the arcs BD, DC is a fifteenth part of the whole circumference.

∴ chords BD, DC are two consecutive sides of a regular quindecagon. By laying out chords consecutively, each equal to BD or DC, we obtain a regular quindecagon inscribed in the circle.

Note. If we draw tangents to the circle at the vertices of an inscribed regular quindecagon, we shall obtain a regular quindecagon circumscribed about the circle.

EXERCISE LXXXIV.

1. Construct angles of 12°, 24°, 6° with the aid of ruler and compasses only.

2 Show how to divide a right angle into 5 equal parts.

3. Construct an isosceles triangle whose vertical angle is three times either angle at the base. [Observe the △ **APC** in Fig. 446.]

4. In Fig. 446, if the circle with centre **A** and radius **AC** cuts the circumcircle of the △ **APC** again in **D** prove that

(i) **CD**=**BC**, and each is a side of a regular decagon inscribed in the circle drawn with centre **A** and radius **AC**,

(ii) **AP**, **PC**, **CD** are sides of a regular pentagon inscribed in the circle passing through **A**, **P**, **C**,

(iii) the circumcircles of △s **APC**, **ABC** are equal.

5 If **ABC** be an isosceles triangle with each of the base ∠s **B** and **C**, double ∠**A**, prove that $\frac{BC}{AB}=\frac{\sqrt{5}-1}{2}$.

(See Ex. 5 (LXXXIII).

6 In Fig 446, if **D** is the mid-point of **BC** prove that $\frac{AD}{AB}=\frac{\sqrt{10+2\sqrt{5}}}{4}$.

7. Prove that if r is the radius of a circle, and a the length of a side of a regular decagon inscribed in it, then $a=\frac{\sqrt{5}-1}{2}r$

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